INTERFEOMETRY APERTURE SYNTHESIS AND RADIO MAPPING

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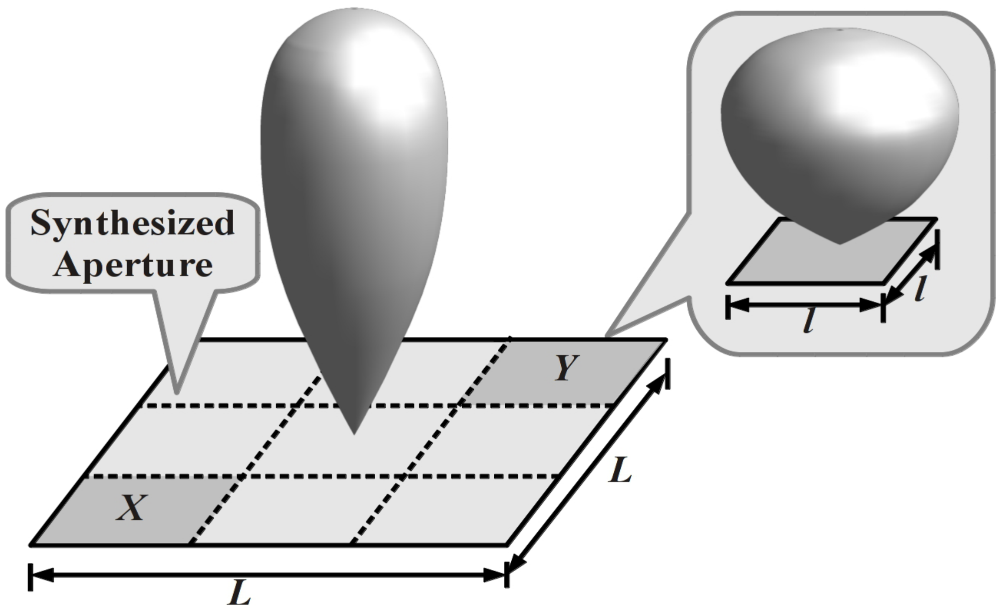
License

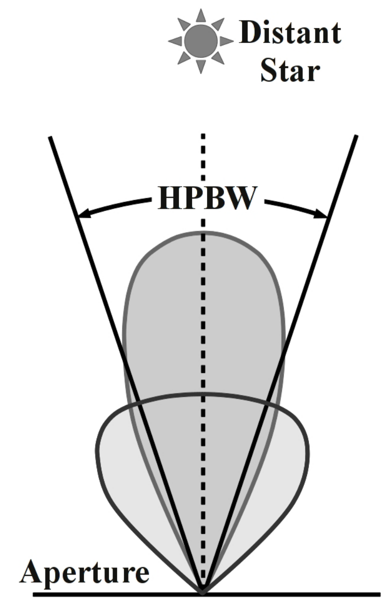
Introduction

In earlier chapters, we have seen some techniques for receiving data and signal processing. Choice for any of these techniques is made based on our final goal. For example, spectral line observations are made to find elements, velocity of the object, etc. A pulsar data on the other hand may be used for analyzing the timing, magnetic field, intensity of the pulsar etc. The interferometer continuum data can be used for creating an image map of the source, which is a two dimensional (right ascension and declination) image of the source intensity distribution. Here we introduce some basic techniques of gathering interferometer array data for producing radio maps.

In earlier days of radio astronomy, a single antenna was multiplexed in time and space to synthesize a large aperture. The process of synthesizing a large antenna aperture using small antennas is known as aperture synthesis. In later stages, the natural rotation of Earth was used to move the antennas in time and space with respect to the observed radio source. Interferometer arrays improved the resolution and reduced observing time. In 1955, the first synthesis array designed to use the Earth's rotation was commissioned. In 1946 Ryle and Vonberg published first interferometric astronomical measurements at radio wavelengths. However, it is also said that Joseph Pawsey were the first to make interferometric measurements.

Synthesizing a Large Ant. with Small Ants.





A large aperture of size $L \times L$ is to be synthesized using some small apertures of size $l \times l$. The large aperture has been broken into nine equal parts of size $l \times l$. Instead of nine, only two fraction apertures X and Y are used for the synthesis. X is kept at a fixed location and Y is multiplexed in time and space. Finally, the data is properly arranged for the nine ($N = 9$) fractions and then added. This gives an equivalent data of the synthesized aperture.

$i_n \, e^{j\phi_n}$ - Current delivered by n^{th} element, i_n – Magnitude, φ_n - Phase

Current magnitudes of all elements are same ($i_n = i_0$). Resulting current contribution from all elements is:

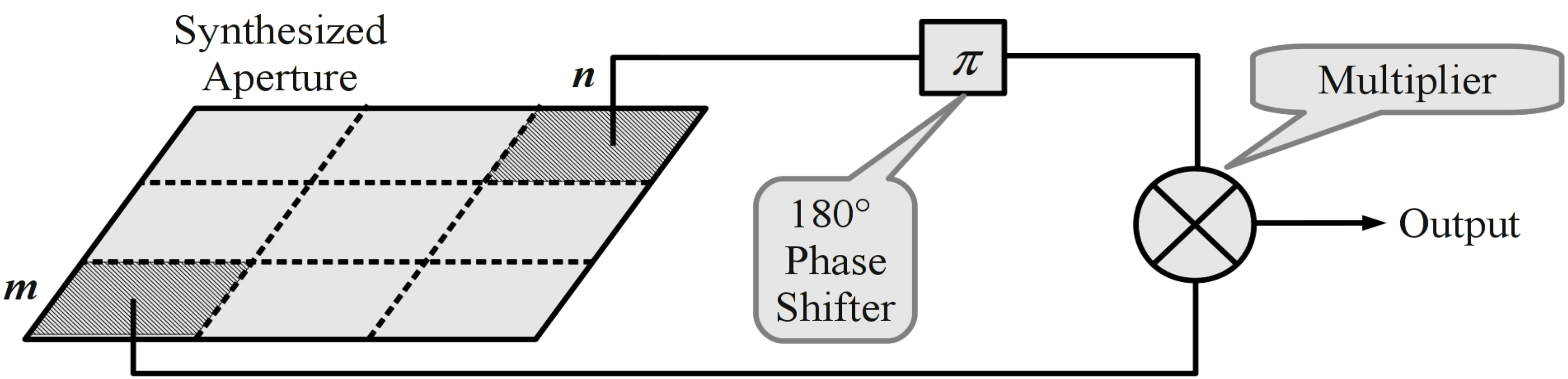
$$i_{\text{total}} = \sum_{n=1}^{N} i_n \, e^{j\phi_n} = i_0 \sum_{n=1}^{N} e^{j\phi_n} \quad \dots (1)$$

Synthesizing a Large Ant. with Small Ants.

Power P delivered to the receiver is:

$$P = k_1 \, i_0^2 \sum_{n=1}^{N} e^{j2\phi_n} \qquad \text{... (2)}$$

Before data addition, phase corrections are made using the data from the fixed aperture which is the reference aperture. This is necessary since the wave-front varies with time. It is also necessary for the radio source under observation to lie within the HPBW of the synthesized aperture as shown:

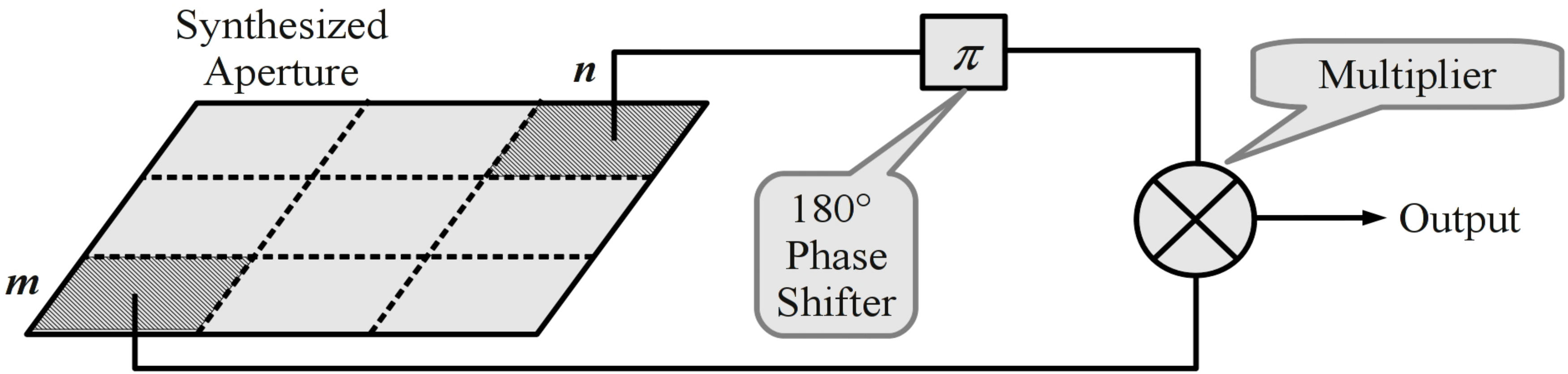


It can find the phases of other elements with reference to a fixed element. Here, the output of the fixed element m is multiplied with the 180° phase shifted output of any another element n. The path lengths from any element to the multiplier are considered identical.

$v\, e^{j\phi_m}$ - Voltages of the m element.

$v\, e^{j\phi_n}$ - Voltages of the n element.

Synthesizing a Large Ant. with Small Ants.



Multiplier output F is a measure of phase difference between the two:

$$F = v^2 e^{j(\phi_m - \phi_n)} \qquad \text{... (3)}$$

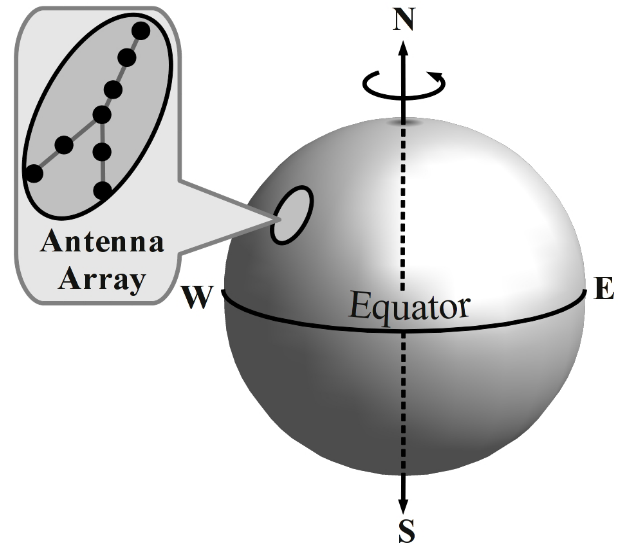
F can be suitably combined with data obtained from each location n to correct the respective phases before addition. In this way, the entire aperture is synthesized.

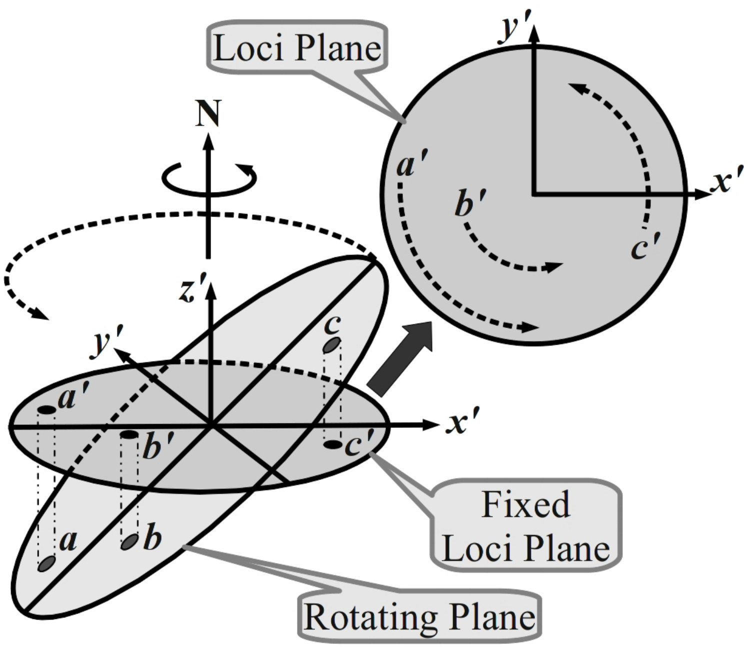
The early days of radio astronomy used this kind of methods. Angular resolutions of 45' at 7.9 m and about 25' at 1.7 m wavelengths were achieved and nearly 5000 radio sources were resolved. The movable antenna was set on railway tracks.

Super-Synthesis using Earth Rotation

We already had an introduction about super-synthesis. We further develop this idea.

Consider an antenna array spread over a plane area at some latitude between 20° and 70° on Earth. These antennas are tracking a distant radio source on the celestial sphere. As seen from the radio source, the entire array appears rotating with Earth.



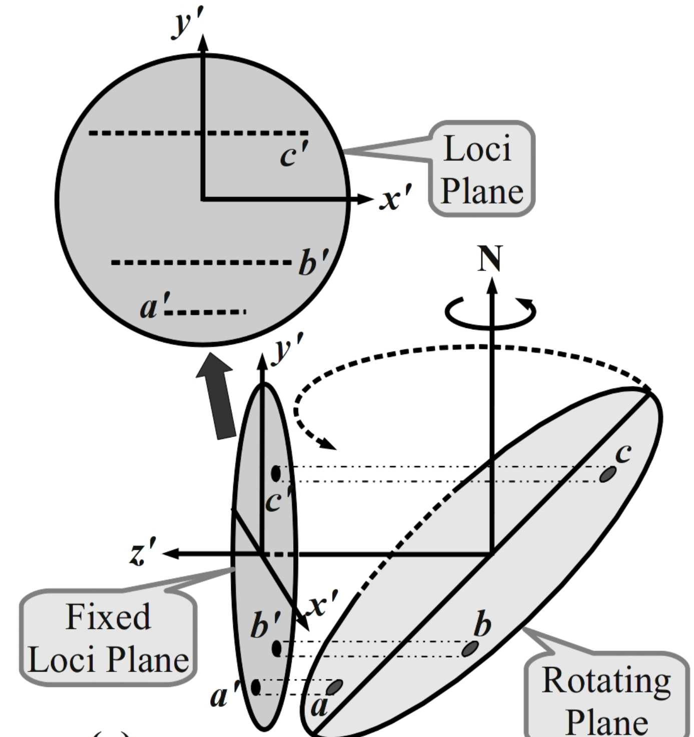


Consider a rectangular coordinate system (x', y', z') with its z' axis pointed to the source. Let the center of the antenna plane be located at its origin. Let the x' and y' coordinates be stationary to the source. As seen from the source, the position of the antennas move over x'-y' plane due to Earth rotation. Consider three antennas a, b and c whose projections on the x'-y' plane will be investigated for the following three cases:

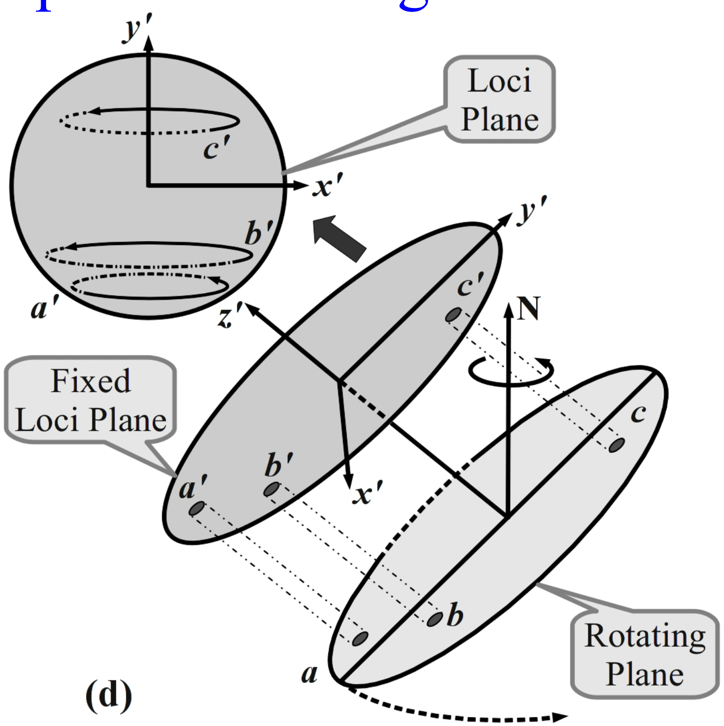
***Case* (*i*): Source on CNP (celestial North pole).** As shown (above), the loci a', b' and c' respectively for the antennas a, b and c will trace circles on the x' - y' plane over a period of 24^h.

Super-Synthesis using Earth Rotation

***Case* (*ii*): Source on the celestial equator.** As shown (right), the loci a', b' and c' trace straight lines on the x' - y' plane. As a special case, if all antennas are posited on a single East-West line, all loci will trace a single straight line and the population on the x' - y' plane will be one dimensional. This is not suitable for making a two dimensional map. Hence for observing on the celestial equator, some antennas should have separations along the North-South axis.



(c)



(d)

***Case* (*iii*): Source in between celestial Equator and CNP.** As shown (left), if the radio source is directed at a celestial latitude other than than 0° and ±90°, the loci will trace ellipses. This follows from the intution that we developed from cases (*i*) and (*ii*).

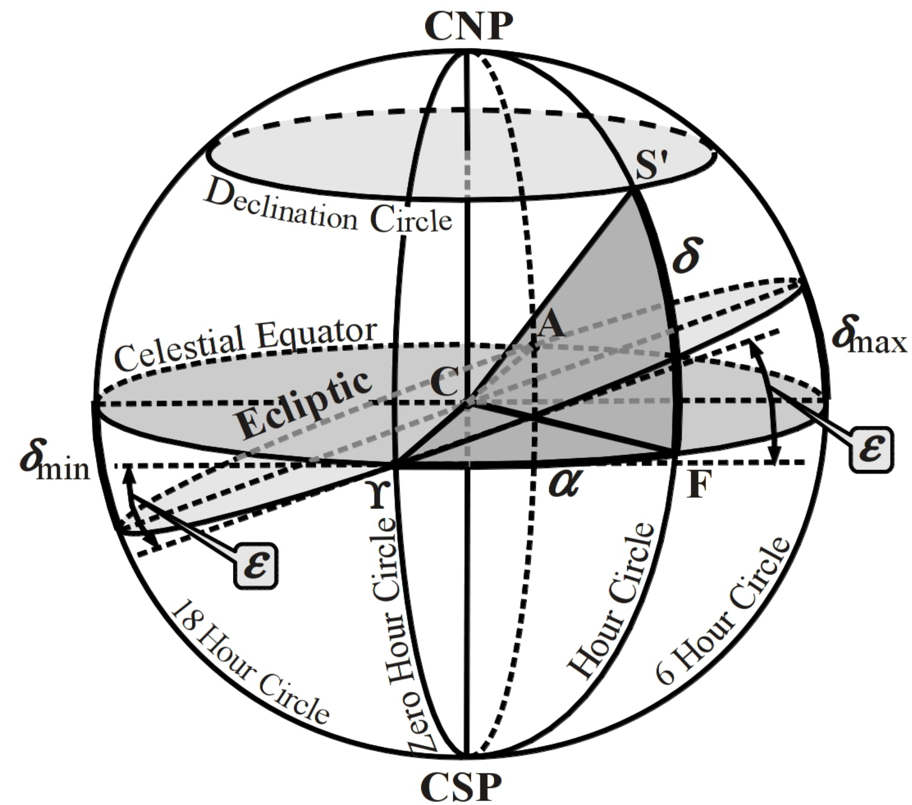
Synthesizing the Aperture

The voltage outputs from individual antennas can be recorded at short intervals of time. These values can be placed on the $x' - y'$ plane on the loci of the antenna. At very short intervals of time, we may record the voltage outputs of individual antennas and place these values on the $x' - y'$ plane. The more is the observation time and the number of antennas in the array, more is the $x' - y'$ plane populated. After 24^{h}, the $x' - y'$ plane will be highly populated with the voltage values resulting from the radio source. This populated area on the $x' - y'$ represents the aperture of a large synthesized antenna.

A single interferometer with adjustable baselines can be used for aperture synthesis aided by Earth rotation. One of the antenna elements is used for phase reference. A clear understanding of equatorial systems, (u,v,w) coordinate system, and (X,Y,Z) coordinate system are required. We begin by visualizing these coordinates together in relation to a radio source and their inter relationships.

Universal Equatorial System (α, δ)

The fundamental great circle is the celestial equator. The point at which the ecliptic crosses the equator from South to North is the ***vernal equinox*** ♈ which is the reference point. The ecliptic is inclined to the celestial equator by 23.5° known as ***obliquity of the ecliptic*** ε. The opposite point of ♈ is the ***autumnal equinox***. Right ascension α and declination δ are respectively measured from ♈ and the celestial equator. Half way between the equinoxes, the Sun's declinations are $\delta_{max} = \varepsilon$ and $\delta_{min} = -\varepsilon$ which are summer and winter solstices.



(α,δ) are used for preparing permanent star charts.

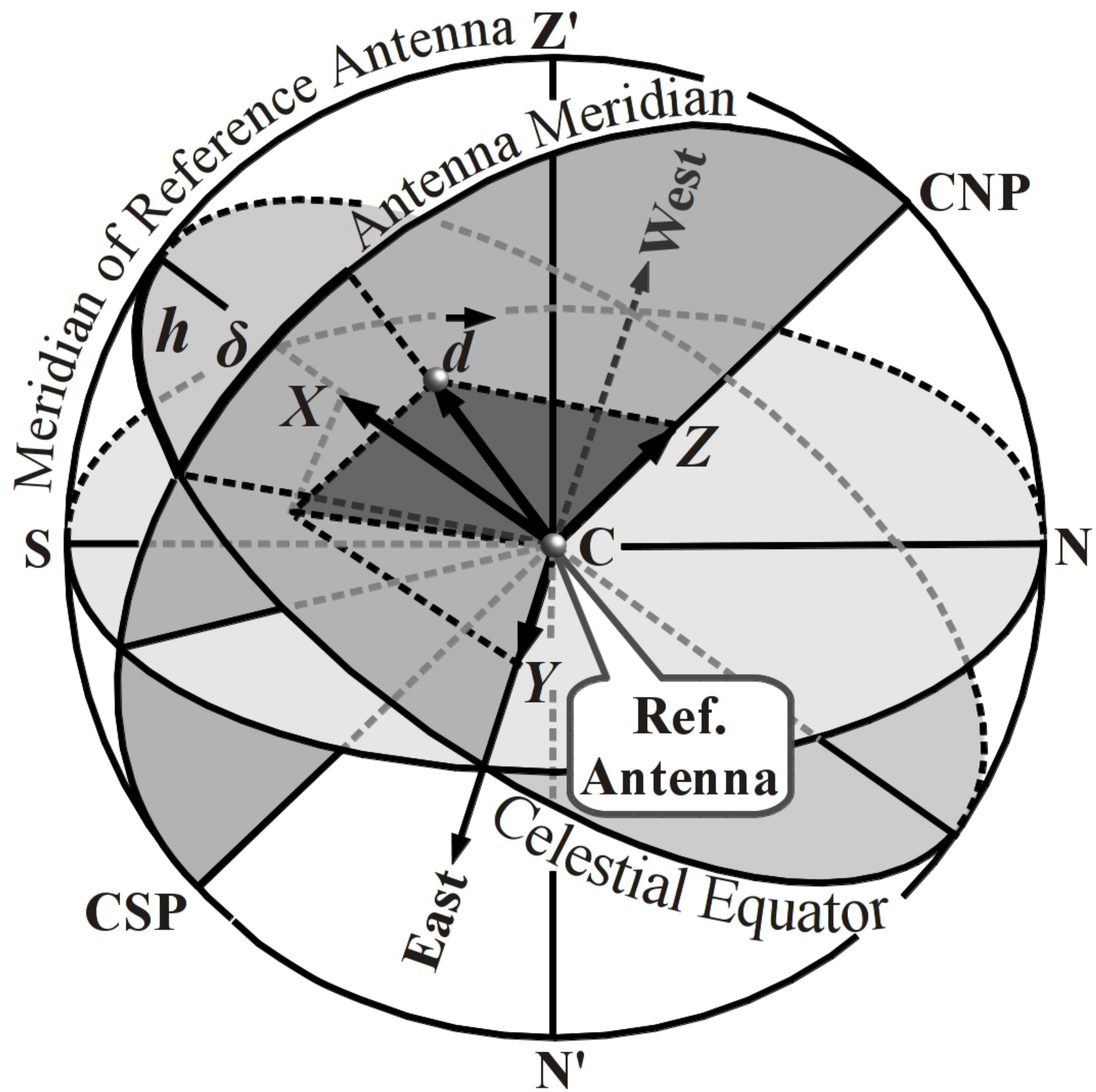
♈ changes with time due to the precession of Earth axis. It is roughly 50.26" per year towards the West. Hence, astronomers also specify a date at which ♈ is considered for a given right ascension and declination of a star. This is known as epoch. For example, the epoch J2000.0 refers to the objects position at Greenwich noon (UT) of 1st January 2000.

X,Y,Z Coordinate System

A reference antenna is the origin C. Position of all other antennas are determined as X, Y and Z. $\vec{d}$ is position vector (***baseline vector***) from reference antenna to any other antenna making an hour angle h and declination δ.

$$d = \sqrt{X^2 + Y^2 + Z^2}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = d \begin{bmatrix} \cos\delta \, \cos h \\ -\cos\delta \, \sin h \\ \sin\delta \end{bmatrix}$$



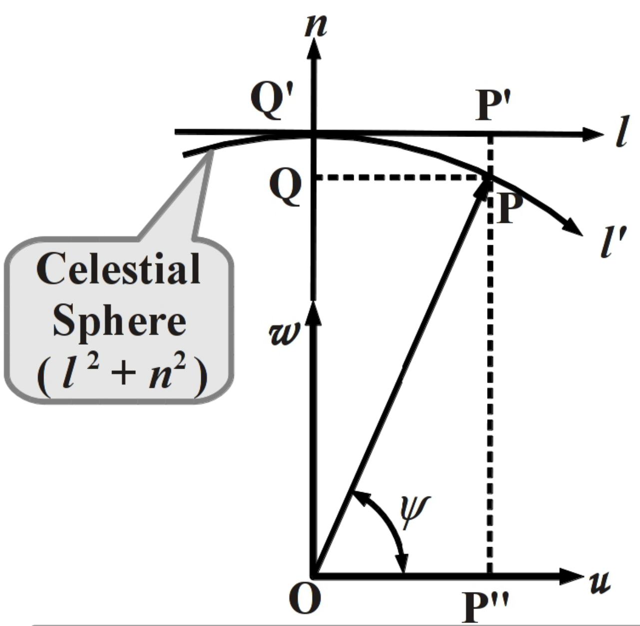
Used for determining relative the positions of antennas within an antenna array of a radio telescope.

l,m,n Coordinate Systems-I

The radio maps constructed for an astronomical objects is an electric field distribution produced by the object on the celestial sphere. These are intensity contours plotted using astronomical coordinates.

For a two dimensional geometry shown, the celestial sphere is denoted by an arc l'. Let the point Q' on the celestial sphere be observed such that source extent on the right side is P and the angle subtended is ψ. A tangent on the celestial sphere at Q' is the coordinate l. The point P is now represented by direction cosines. Hence,

$$l = \mathrm{Q'P'} = \mathrm{OP''} = \cos\psi \quad \text{and} \quad n = \mathrm{OQ} = \sin\psi$$



l = Q'P'' = OP'' = cos ψ
n = OQ = sin ψ
$l'^2 = (\mathrm{OP''})^2 + (\mathrm{P''P})^2 = l^2 + n^2$

Note that the origin of n-axis is O and not Q'. Therefore, $l'^2 = (\mathrm{OP''})^2 + (\mathrm{P''P})^2 = l^2 + n^2$

Hence, any point on the celestial sphere is represented as $l'^2 = l^2 + n^2$

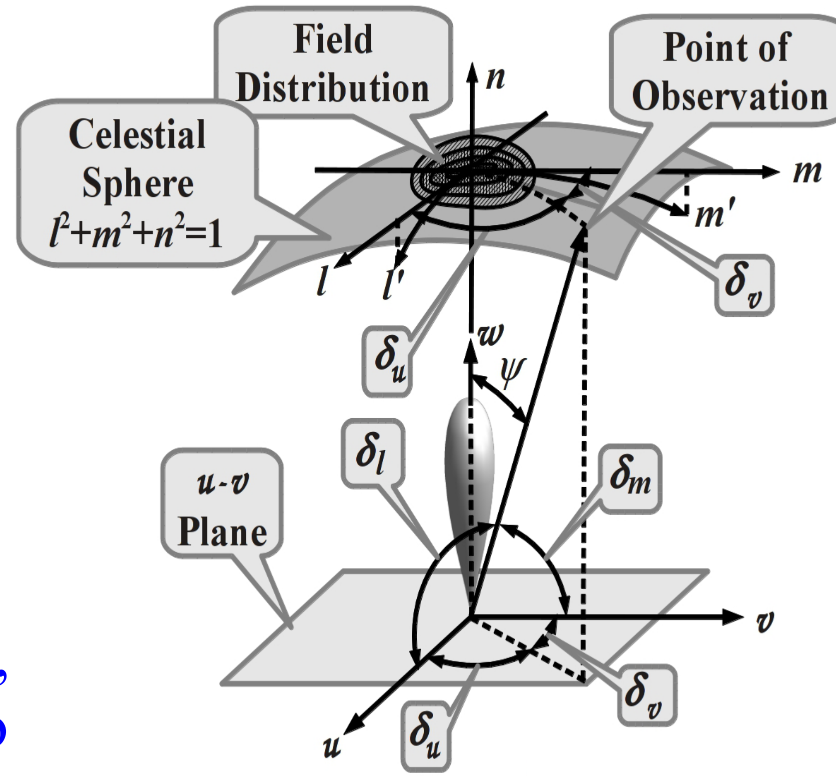
If the source extent is small, $n \approx 0$ and hence, $l' \approx l$.

l,m,n Coordinate Systems-II

Radio telescopes may be thought as a single highly directional antenna with a beam-width much smaller than the source extent, which scans the source to receive and estimate the spatial intensity distribution of the source.

Consider a three dimensional space geometry. The beam scanning is now a function of two angles δ_l and δ_m. The direction cosines are now l, m and n as shown. The arcs are represented by l' and m'. Any point on the celestial sphere can be represented by l, m and n as:

$$\left\{ \begin{array}{c} n = \cos\psi \\ l = \sin\psi\ \cos\delta_u = \cos\delta_l \\ m = \sin\psi\ \cos\delta_v = \cos\delta_m \end{array} \right\}$$

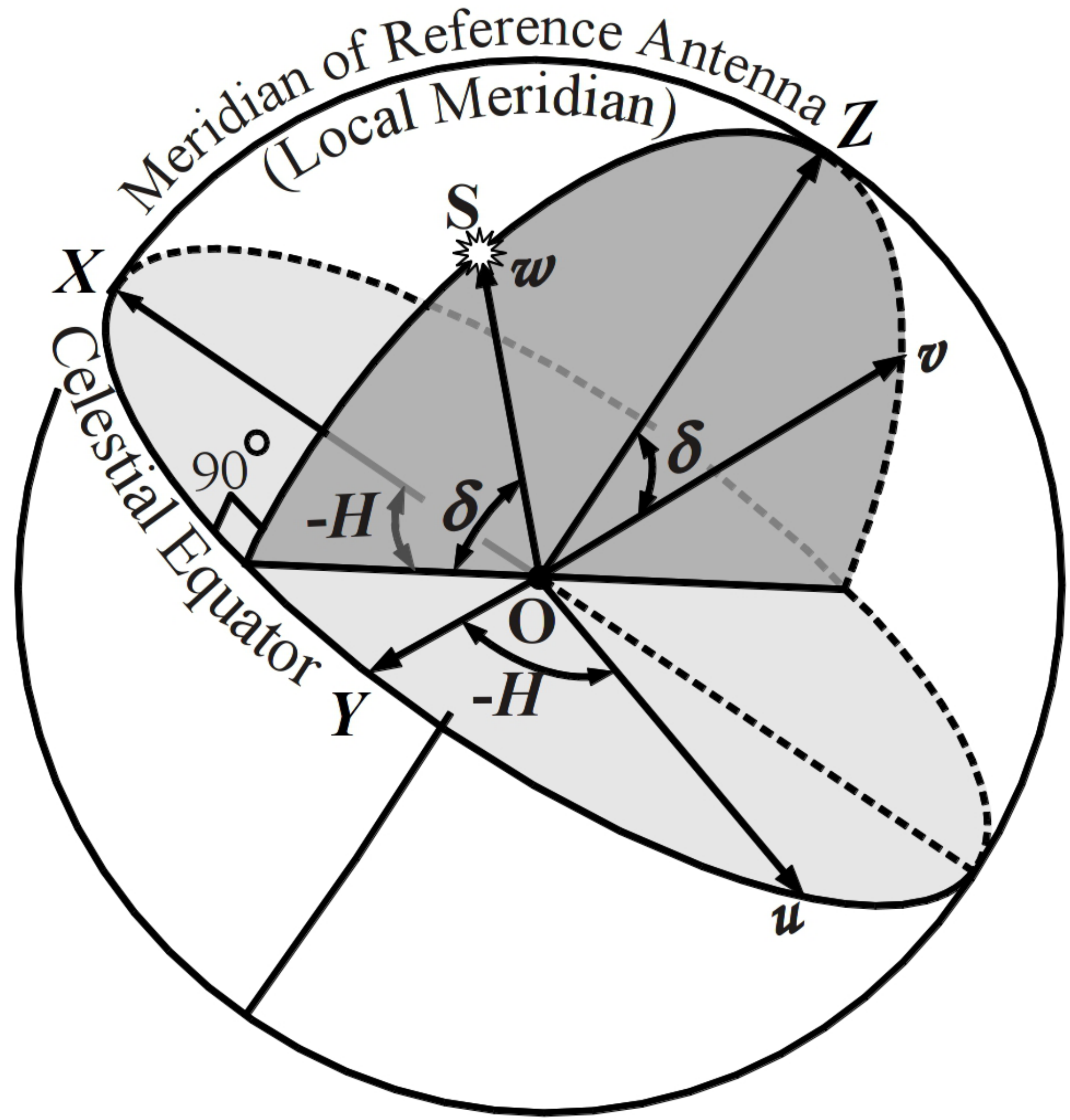


The geometrical relationships between the l, m, n are such that their squares always add to unity as $l^2 + m^2 + n^2 = 1$

Celestial sphere is spherical. A very small surface of the celestial sphere may be approximated as a flat surface. It assumes $n \approx 0$. Under such conditions, radio astronomers use an approximation: $\left\{ \begin{array}{c} l \simeq l' \\ m \simeq m' \end{array} \right\}$ when $n \approx 0$

u, v, w and X, Y, Z Coordinate Systems

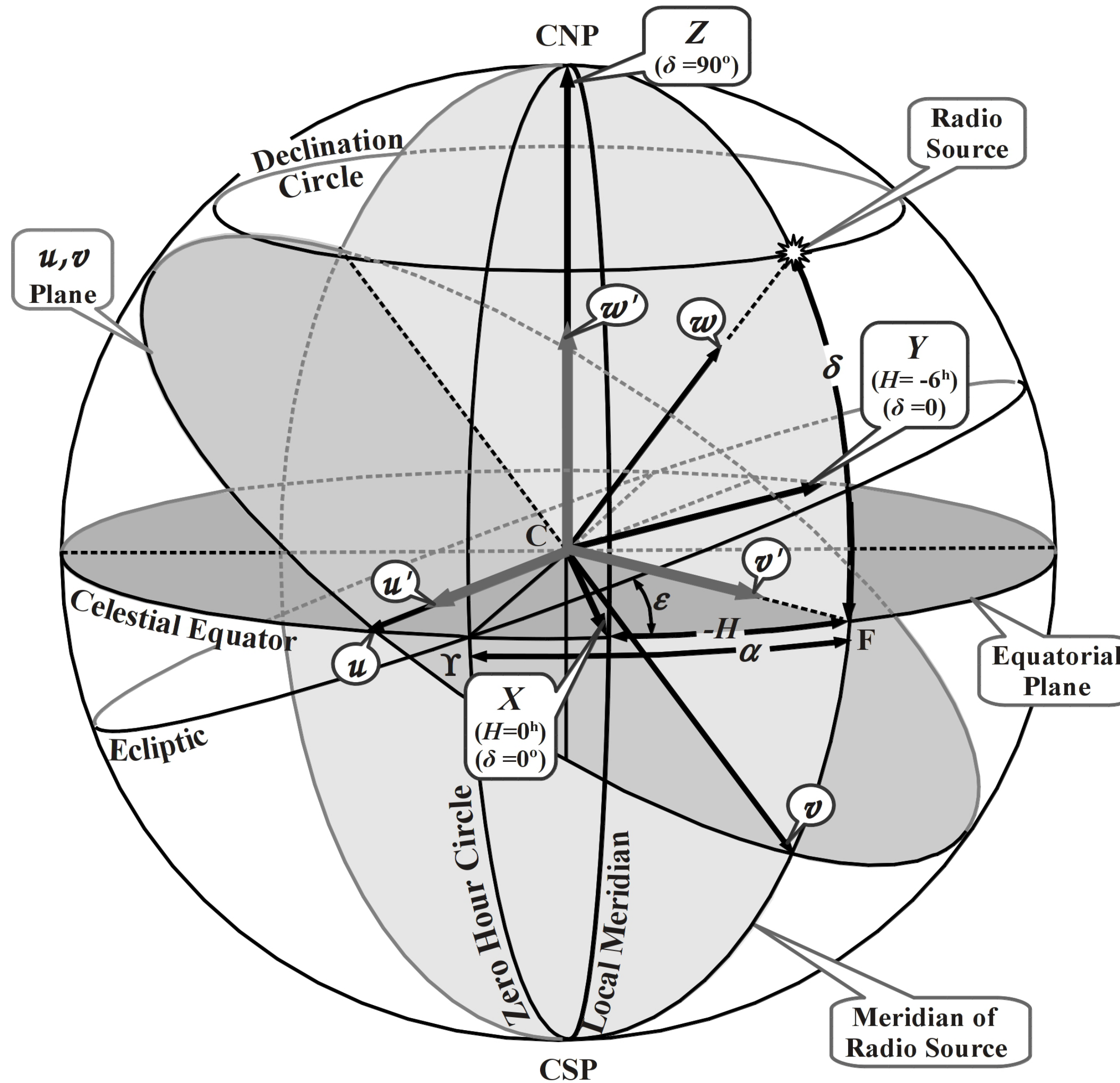
The w coordinate is always directed to the source **S**, and u always lies on the equatorial plane on the eastern side of **S**. The u–v plane is always perpendicular to the source direction w. It keeps its orientation fixed with respect to source. H is local hour angle of the phase reference position on the radio source with respect to local meridian. Here, **S** is in eastern hemisphere and so H is negative. The X-Y plane sits on the plane of the celestial equator such that the X-axis lies on the meridian plane and Y-axis points towards the East. The Z-axis points towards North.



The u,v,w coordinates are used by radio interferometers for super-synthesis in radio astronomy.

All Coordinates of a Radio Source

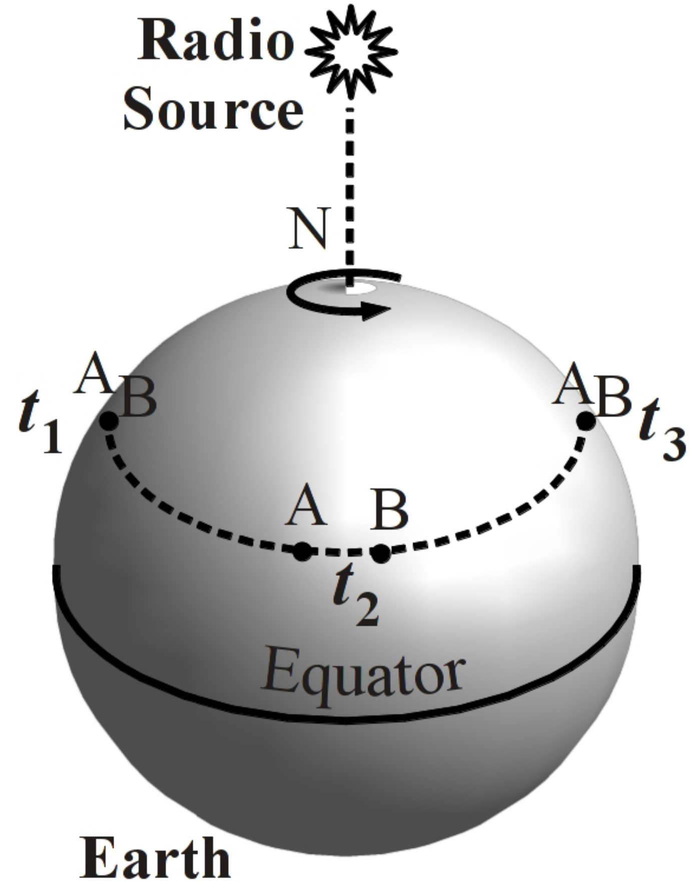
The Z axis of the right handed X,Y,Z coordinate points to the CNP. The Y axis faces East and X axis lies in the local meridian plane. ♈ is the vernal equinox and ε is is the obliquity. The w axis points to the source. The v axis is in a longitudinal plane touching the two celestial poles, and the direction of observation. Right ascension α and declination δ are marked. Hour angle H w.r.t. the local meridian is shown. The u',v',w' coordinates is a special case of u,v,w coordinates when source is at the CNP.

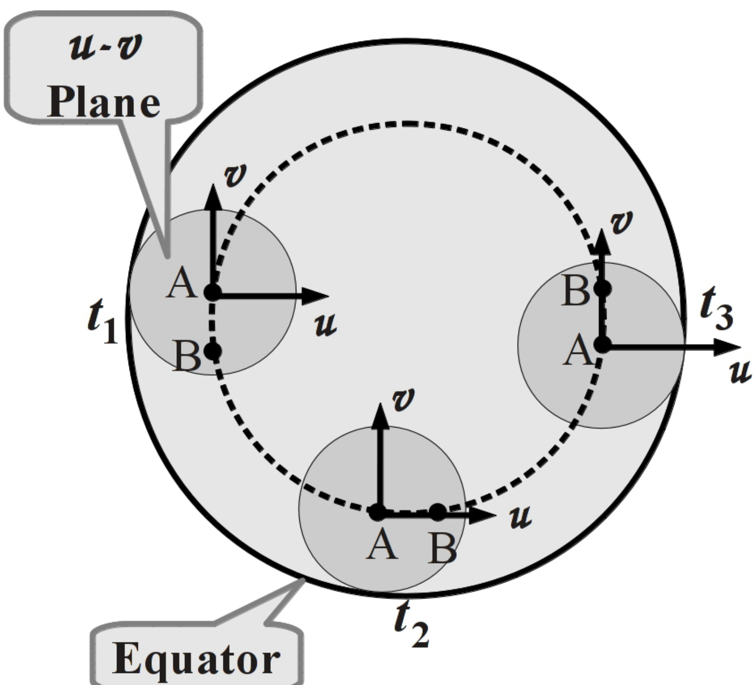


Note: As α and δ of the source change with time, the u, v and w also change their orientations accordingly as seen by the observer. However, orientation of u and v does not change as seen from the source.

Interferometer with Source at a Pole-I

Consider a single interferometer having two antennas of equal diameter D. These are positioned on a East-West line having same latitude and separated by a distance d as shown. Let the source be towards the CNP. The projection $d\cos\theta$ of the baseline d on the u-v plane rotate with Earth rotation. The antenna positions for three time instants (t_1, t_2 and t_3) are shown on right.

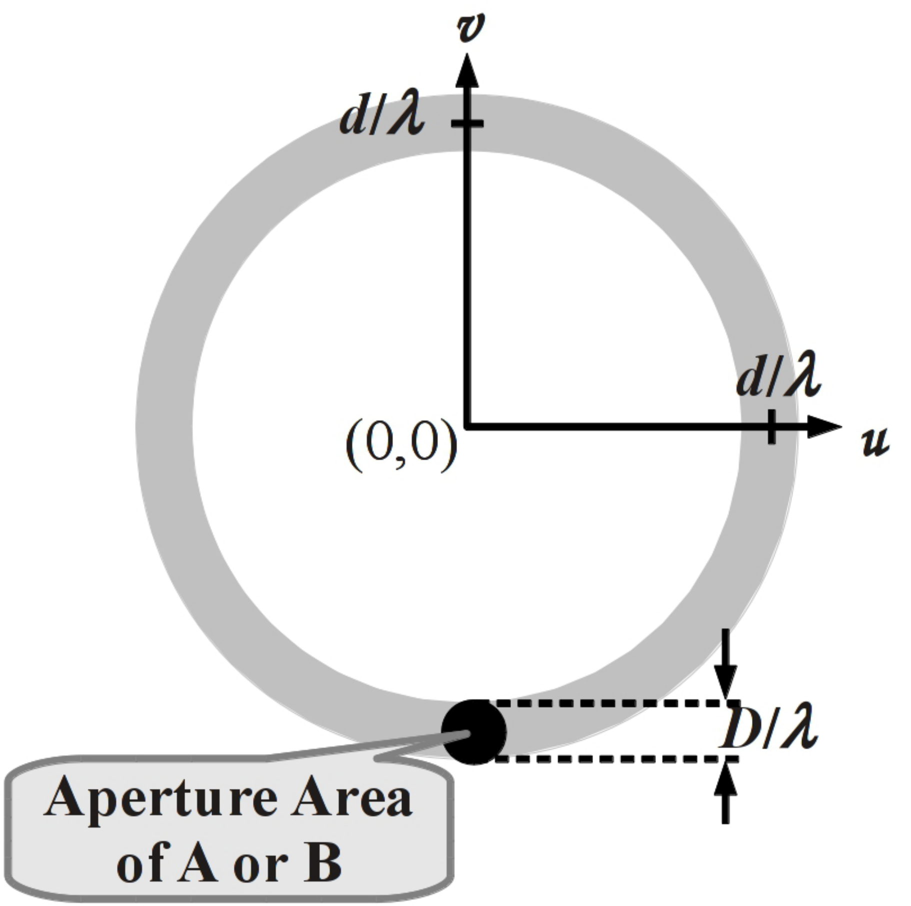




The u-v plane is parallel to the equatorial plane since the former is always perpendicular to the source direction. If we center the u-v plane on antenna A, the position of antenna B for t_1, t_2 and t_3 can be seen as illustrated on left.

Interferometer with Source at a Pole-II

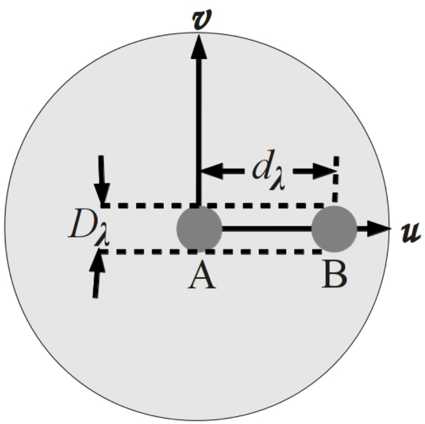
For a twelve hours observation, the trace of antenna B produces a half circular ring of thickness D/λ on the u-v plane. If we now center the u-v plane on antenna B, the trace of antenna A produces another half circular ring of thickness D/λ on the other side of the u-v plane. We can join the two halves to form a complete ring of thickness D/λ in the u-v plane. This is shown on right. For this particular case, the projected baseline is identical to the actual baseline d since the source is towards the CNP. On the u-v plane, the central radius of the ring is projected as $u = d/\lambda$ and $v = d/\lambda$, where λ is the wavelength of observation.

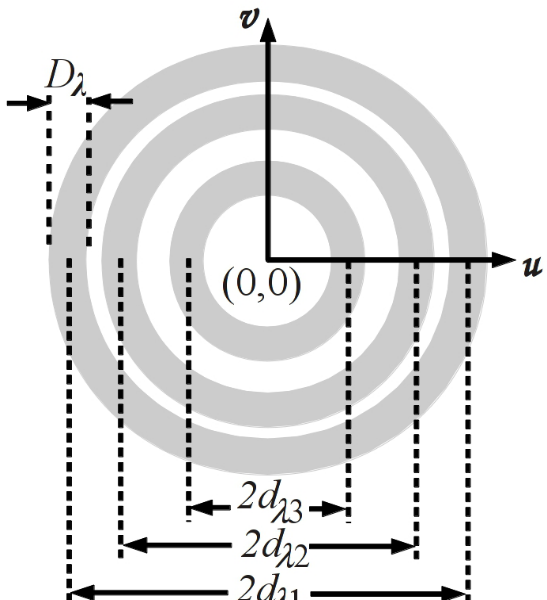


Populating *u-v* by repositioning Antenna-I

The synthesized aperture should lie on *u-v* plane for making a radio map of the source. Reducing distance between the two antennas reduces the baseline resulting in smaller circles on the *u-v* plane. Hence by several twelve hour observations with different baselines $d_{\lambda n} = d_n/\lambda$, the *u-v* plane can be populated well. Here, d_n represents the distance between the two antennas, $n = 1, 2, 3, \ldots$ represents cases of different baselines and λ is wavelength of observation.

Consider an interferometer formed from two equal diameter (D) antennas A and B. As shown on right, let *u-v* coordinates be centered on A. Antenna A is the reference antenna.



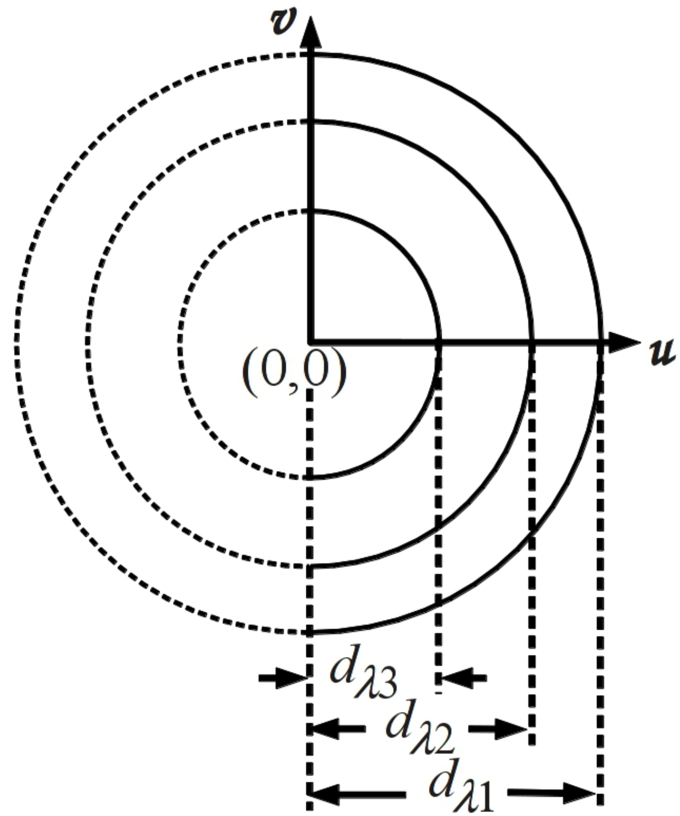


With Earth rotation, one half circle is traced on the *u-v* plane every twelve hours. The semi-circular ring has a radius equal to the baseline d and thickness $D_\lambda = D/\lambda$. The other half of the ring is constructed using the relation: $\mathcal{V}(-u,-v) = \mathcal{V}^*(u,v)$

A complete circular ring is obtained by joining the two halves. The *u-v* plane coverage for three different baselines $d_{\lambda1}$, $d_{\lambda2}$ and $d_{\lambda3}$ is shown on left.

Populating *u-v* by repositioning Antenna-II

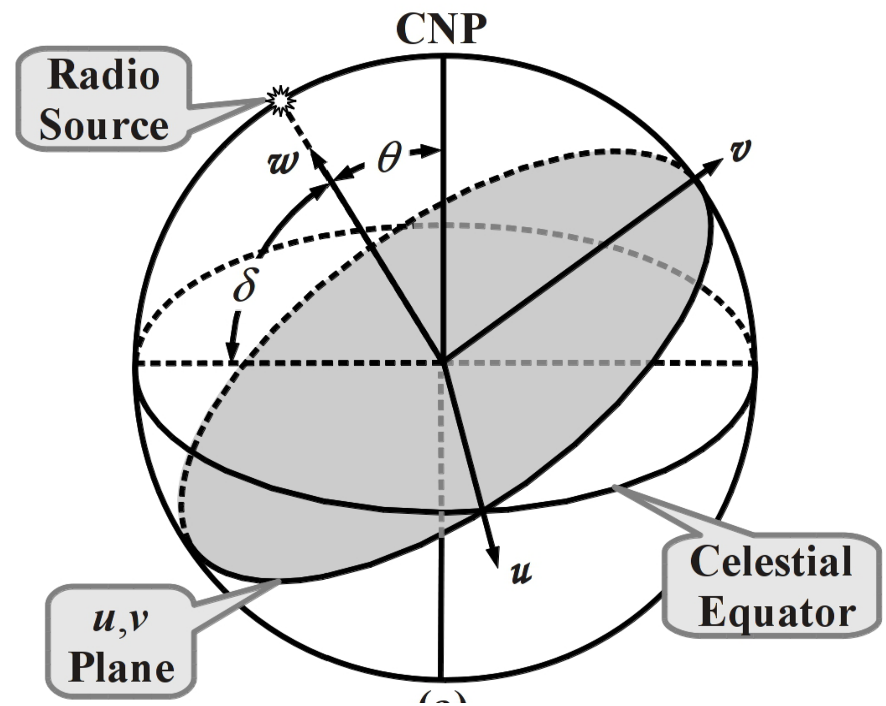
Thus by changing baselines, it is theoretically possible to populate the entire *u-v* plane. In practice, the thickness of the rings are zero as shown on right. The fault lies with the antenna output which converts the entire dish response to a single output. Each visibility data contains only a resultant information produced from (*i*) a band of spatial frequencies determined by the aperture diameter of individual antennas, and (*ii*) its distance from the center of the *u-v* plane (projection of the baseline). Hence synthesized aperture in the *u-v* plane is only an approximation.

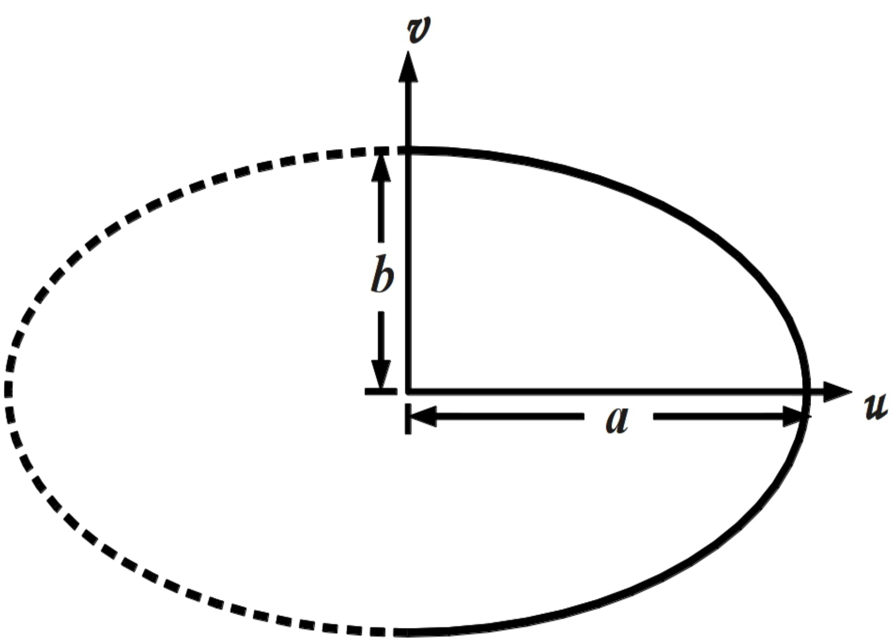


Further, the maximum diameter of the rings are restricted by the maximum possible physical distance between the two antennas. Hence, above certain diameter in the *u-v* plane, there is no data. Similarly, the central portion of the *u-v* plane also remains empty which is determined from the shortest baseline. The shortest baseline possible is *D* since two antennas can not physically accommodate themselves less than this dimension.

Baseline parallel to Celestial Equator-I

Consider an interferometer having a East-West baseline formed by antennas A and B, with A as reference antenna. Hence the baseline will be parallel to the celestial equator. Let the observed source be inclined at an angle θ from the CNP as shown on right.



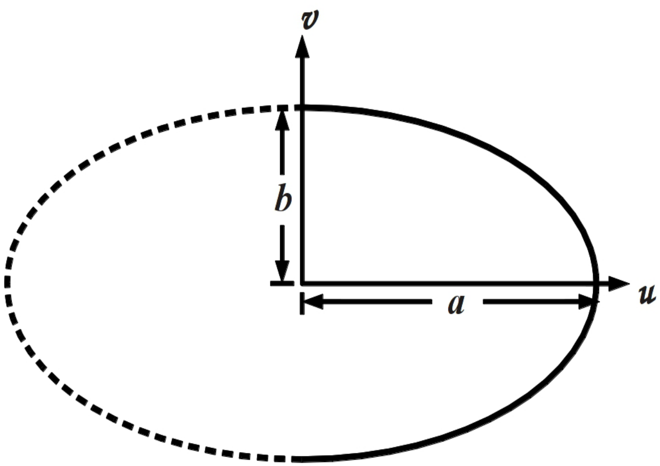


$a = d/\lambda$

$b = d \cos\theta / \lambda = d \sin\delta / \lambda$

As shown on left, antenna B traces an ellipse on the u-v plane with Earth rotation. In twelve hours, half of the ellipse (shown by thick line) is formed. Using the visibility relation $\mathcal{V}(-u,-v) = \mathcal{V}^*(u,v)$, the other half (shown by dotted lines) is generated from the data.

Baseline parallel to Celestial Equator-II



$a = d/\lambda$

$b = d\cos\theta/\lambda = d\sin\delta/\lambda$

The major and minor axis of the ellipse respectively represented by a and b are given as:

$$a = \frac{d}{\lambda} \quad \dots (1)$$

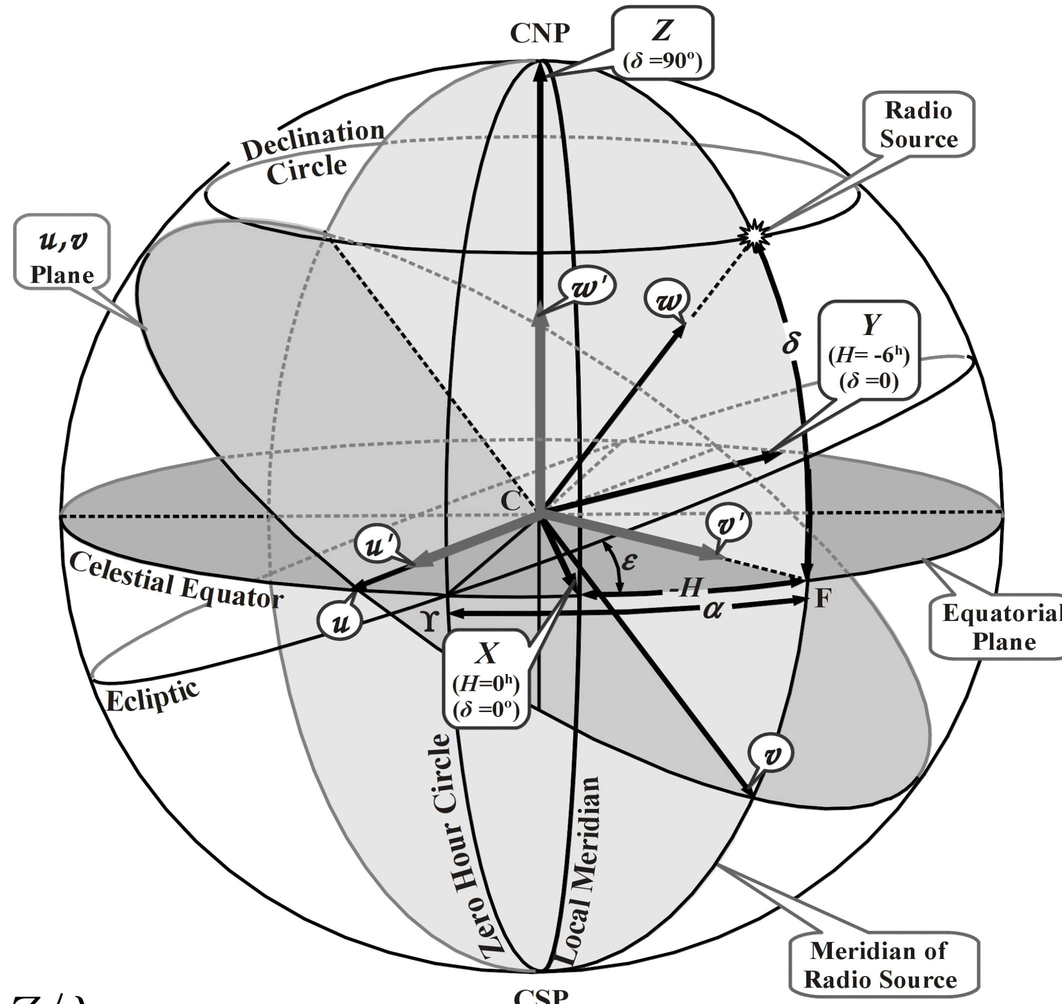
$$b = \frac{d\ \cos\theta}{\lambda} = \frac{d\ \sin\delta}{\lambda} \quad \dots (2)$$

Here, d is the baseline (distance between the antennas), is the inclination of the source from CNP, is the declination of the radio source and is the wavelength.

As a special case, when the source is on the celestial equator ($\theta = 90°$), Eq. (2) gives $b = 0$. Hence, antenna B traces a straight line on the u-v plane. For any source below and above the celestial equator, antenna B traces an ellipse on the u-v plane. Since the baseline used here is parallel to the celestial equator, the center of the ellipse coincides with that of the u-v plane. This is true for any source declination.

Baseline component orthogonal to CEq.-I

If antennas are not exactly on the East-West line, then the baseline has a component along North-South line. The ellipse splits and their centers keep changing with declination angle. To understand this first see the X,Y,Z coordinates. The local hour angle H and the declination δ are marked. It is used for finding all the baselines when number antennas are more than two. Later, these are transformed into u,v,w coordinates.



The components of a baseline d when expressed in X,Y,Z coordinates are:

$$X_\lambda = X/\lambda \qquad Y_\lambda = Y/\lambda \qquad Z_\lambda = Z/\lambda$$

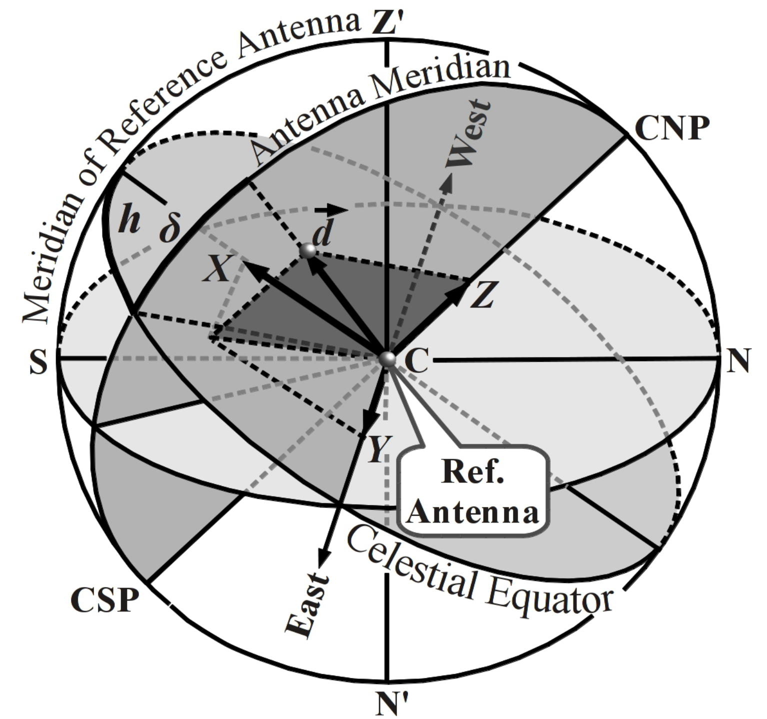
These are but the coordinate differences for any two antennas in the antenna array. The conversion of X,Y,Z to u,v,w coordinates in given by:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \sin H & \cos H & 0 \\ -\sin\delta\cos H & \sin\delta\sin H & \cos\delta \\ \cos\delta\cos H & -\cos\delta\sin H & \sin\delta \end{bmatrix} \begin{bmatrix} X_\lambda \\ Y_\lambda \\ Z_\lambda \end{bmatrix} \quad \ldots (1)$$

Baseline component orthogonal to CEq.-II

In VLBI observations, the X axis is set on Greenwich meridian, and H is measured from there instead from the local one. The quantities H and δ are fixed on the phase reference position. Phase reference position is a point on the source, usually at its center such that the visibility has maximum amplitude under fringe stop condition. The w axis is directed at this point and all measurements are done with this as reference.

Let the antennas be referenced as m and n. With respect to m antenna, h_{mn} and δ_{mn} are respectively hour angle and declination of the n antenna forming the baseline. This is like positioning the reference antenna at the origin of a X,Y,Z coordinate system and measuring h (hour angle) and δ (declination) of the other antenna. In general, the $X_\lambda,Y_\lambda,Z_\lambda$ coordinates of the any n antenna with reference to m antenna are:



$$\begin{bmatrix} X_{\lambda mn} \\ Y_{\lambda mn} \\ Z_{\lambda mn} \end{bmatrix} = d_{\lambda mn} \begin{bmatrix} \cos\delta_{mn}\ \cos h_{mn} \\ -\cos\delta_{mn}\ \sin h_{mn} \\ \sin\delta_{mn} \end{bmatrix} \quad \ldots (2)$$

Here, $d_{\lambda mn}$ is baseline between m and n in number of wavelengths.

Baseline component orthogonal to CEq.-III

By substituting Eq. (2) in Eq. (1) and after simplification we obtain:

$$\begin{bmatrix} u_{mn} \\ v_{mn} \\ w_{mn} \end{bmatrix} = d_{\lambda mn} \begin{bmatrix} \cos\delta_{mn}\ \sin(H - h_{mn}) \\ \sin\delta_{mn}\ \cos\delta - \cos\delta_{mn}\ \sin\delta\ \cos(H - h_{mn}) \\ \sin\delta_{mn}\ \sin\delta + \cos\delta_{mn}\ \cos\delta\ \cos(H - h_{mn}) \end{bmatrix} \quad \ldots (3)$$

Eq. (3) expresses the baseline d_{mn} in terms of u_{mn}, v_{mn} and w_{mn} whose dimensions are in spatial frequency (number of wavelengths).

All the baselines of an interferometer array are brought into the u,v,w domain with the help of Eq. (3). If the telescope antennas use an altazimuth system, it would be preferable to use Eq. (4) for determining the $X_\lambda, Y_\lambda, Z_\lambda$ components.

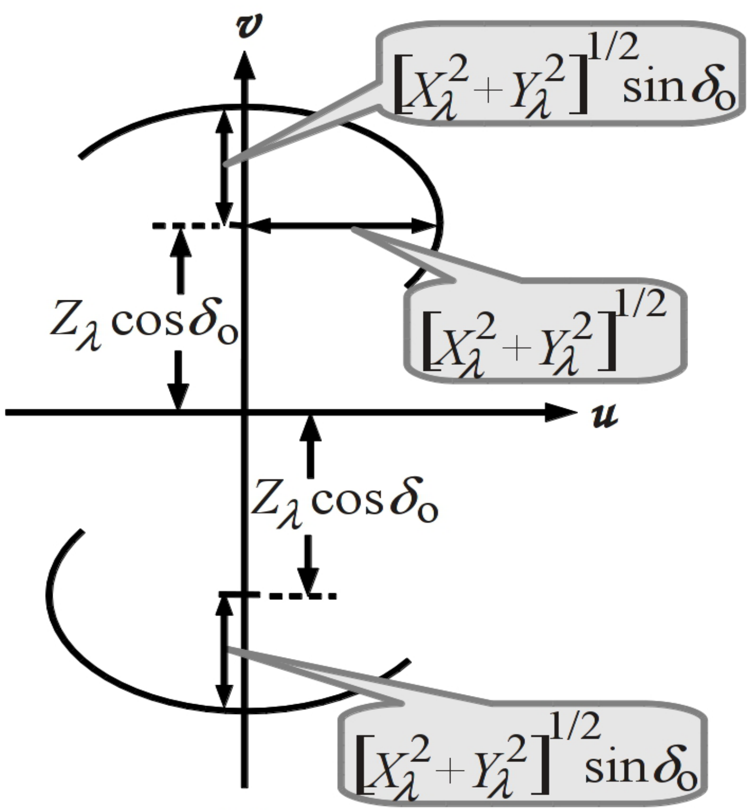
$$\begin{bmatrix} X_{\lambda mn} \\ Y_{\lambda mn} \\ Z_{\lambda mn} \end{bmatrix} = d_{\lambda mn} \begin{bmatrix} \cos\mathcal{L}\ \sin\mathcal{E}_{mn} - \sin\mathcal{L}\ \cos\mathcal{E}_{mn}\ \cos\mathcal{A}_{mn} \\ \cos\mathcal{E}_{mn}\ \sin\mathcal{A}_{mn} \\ \sin\mathcal{L}\ \sin\mathcal{E}_{mn} + \cos\mathcal{L}\ \cos\mathcal{E}_{mn}\ \cos\mathcal{A}_{mn} \end{bmatrix} \quad \ldots (4)$$

Here, $\mathcal{A}_{mn}$ and $\mathcal{E}_{mn}$ respectively represent the altitude and elevation differences between any two antennas, and $\mathcal{L}$ is the latitude of the reference point of the coordinate system (approximately the center of the baseline).

Baseline component orthogonal to CEq.-IV

For a single interferometer, if the phase reference position of the radio source be (H_0, δ_0), then using the the first and second rows of the matrix shown in Eq. (1) one can obtain:

$$u^2 + \left[\frac{v - Z_\lambda \cos\delta_0}{\sin\delta_0}\right]^2 = X_\lambda^2 + Y_\lambda^2 \quad \ldots (5)$$

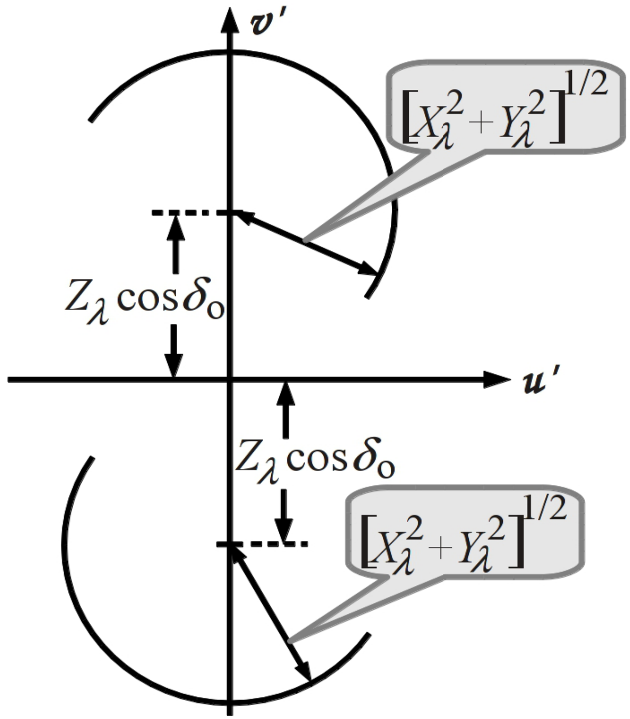


Eq. (5) represents an ellipse. Note that if the component Z_λ is not zero, i.e. if a baseline or its component exist along North-South direction, the ellipse splits into two halves in the *u-v* plane. We may also express Eq. (5) in a general form given as:

$$u_{mn}^2 + \left[\frac{v_{mn} - Z_{\lambda mn} \cos\delta_0}{\sin\delta_0}\right]^2 = X_{\lambda mn}^2 + Y_{\lambda mn}^2 \quad \ldots (6)$$

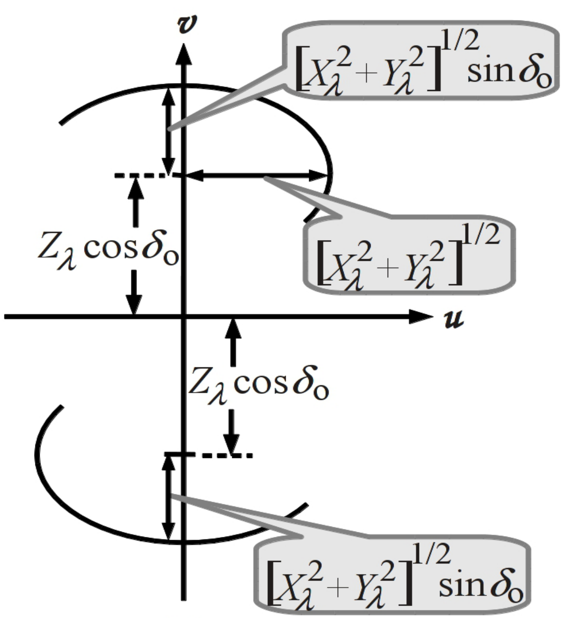
Note that δ_0 remains unchanged for all the interferometers within a local array since the astronomical source is extremely distant.

Baseline component orthogonal to CEq.-V



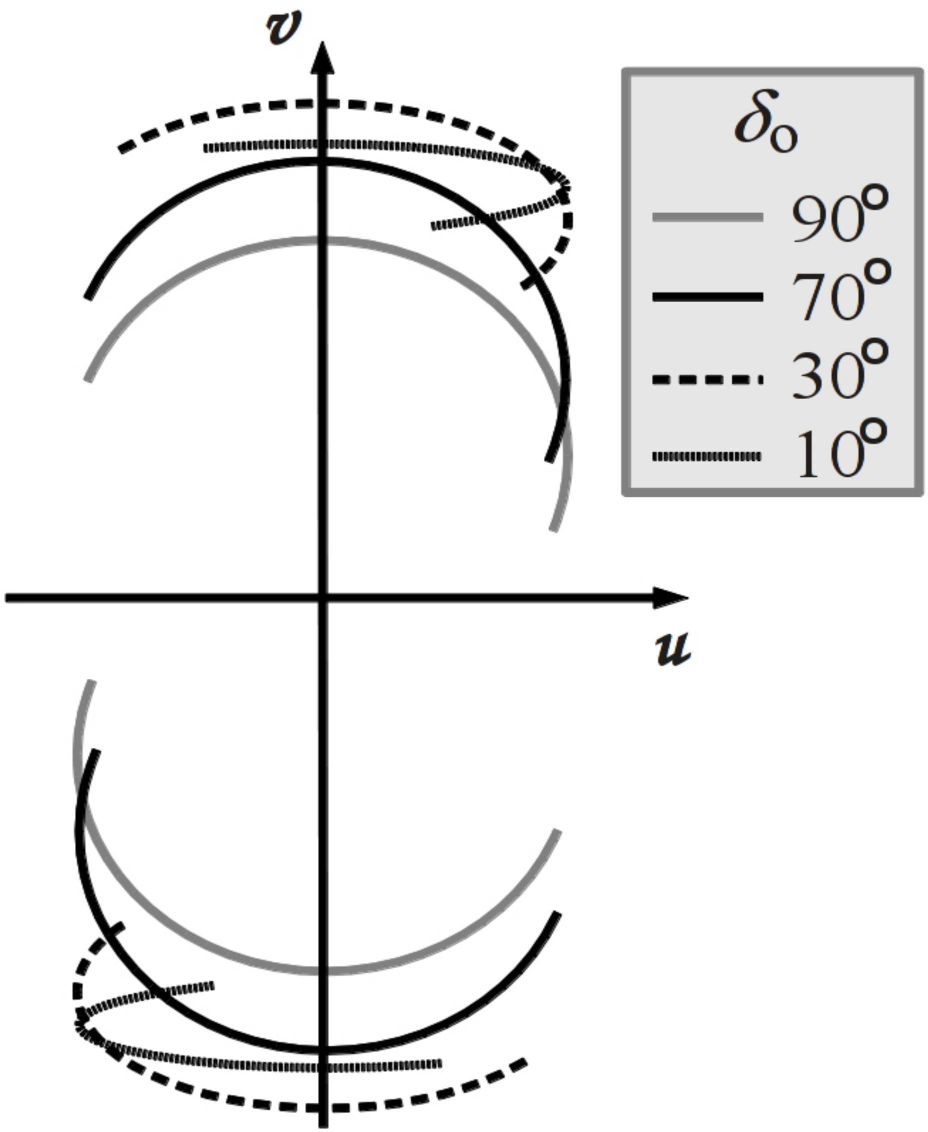
Locus of the ellipse on u'-v' plane for a baseline having a non-zero Z_λ observing a radio source at the CNP. The coordinates used here is u',v' which is a special case of the u,v coordinates. The radius of the split ellipse is constant which is a split circle.

Locus of the ellipse on u-v plane for a baseline having a non-zero Z_λ observing a radio source at declination δ_0 (away from CNP). The major axis of the ellipse stays along u, and its length is same as the radius of the circle of the previous case. The length of the minor axis along v varies with the declination δ_0 of phase reference point. However, the distance between the major axes of the partial ellipses remain unchanged.



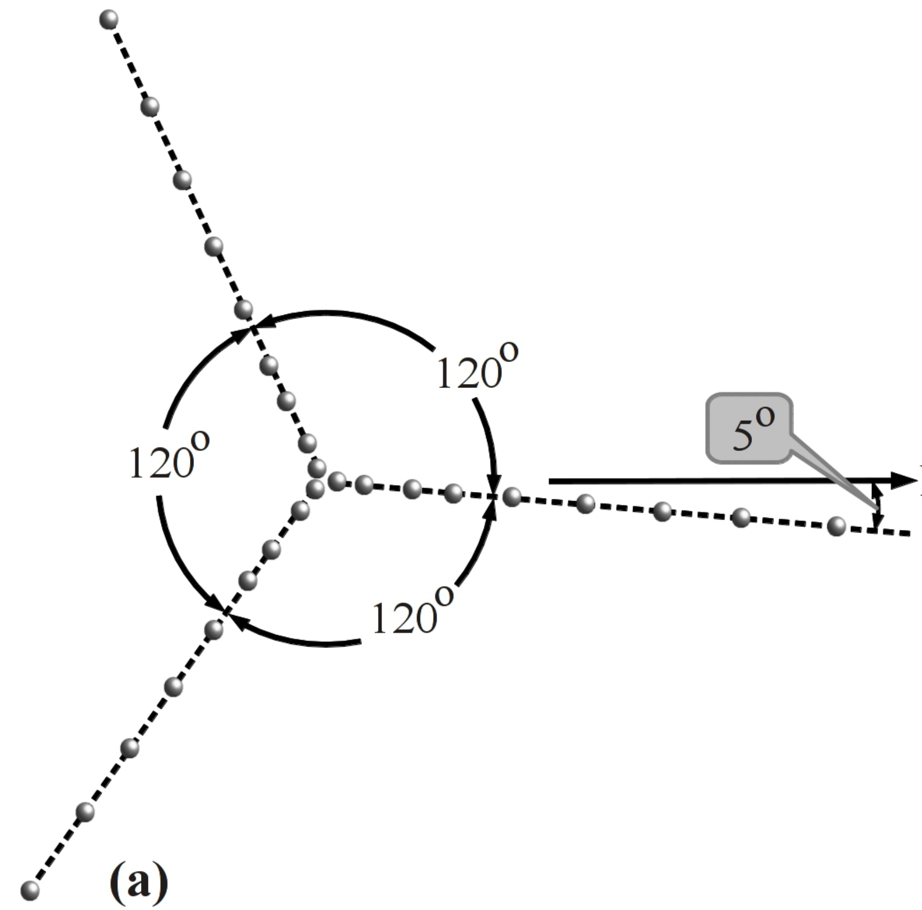
Baseline component orthogonal to CEq.-VI

Tentative plots for different declinations for a baseline having a non-zero Z_λ. As the declination increases, i.e. the more closer the source is towards CNP, the eccentricity of the split ellipses decreases. Finally at $\delta_0 = 90^o$, the arcs resembles split circles. If the Z_λ component becomes zero, the split circle/ellipse join to form a complete circle/ellipse.



Populating the u-v plane using Arrays-I

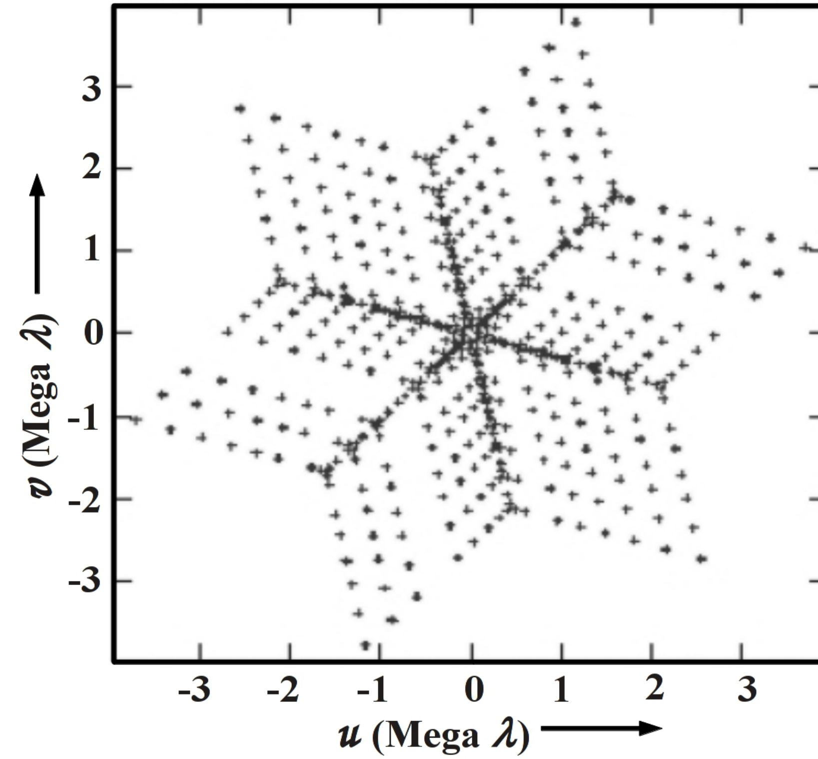
We have seen how an interferometer fills the u-v plane with visibility $\mathcal{V}(u,v)$. We also know that by employing more number of antennas (N), the number of baselines grows to $N(N-1)/2$. A small increase in N results in a large increase of baselines thereby filling the u-v faster, provided each baseline size is unique. A redundancy in baseline size may produce duplicate data set and is of no use. However, baseline redundancy is sometimes useful for phase corrections at the cost of less sensitivity. Some antennas must be aligned along North-South directions for observing radio sources close to the celestial equator, such that the u-v plane is filled in two dimensions.

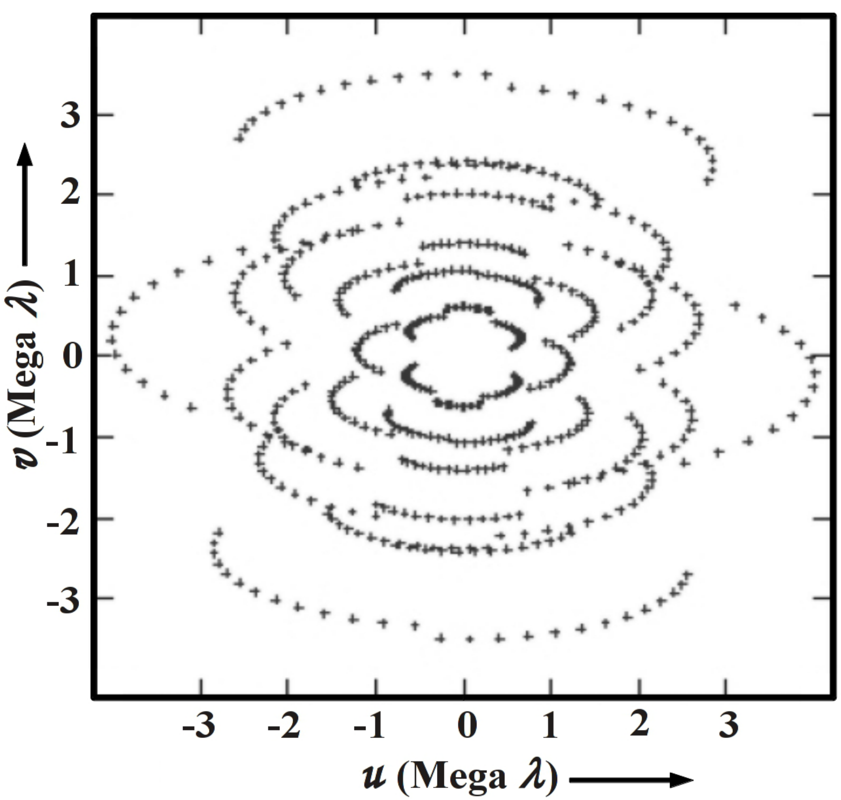


The VLA (very large array, New Mexico) consists of 27 antennas in a Y – configuration.

Populating the *u*-*v* plane using Arrays-II

A star like data set on the *u*-*v* plane is obtained at any instant of time when the VLA antennas observe a source at zenith (see right). With the rotation of Earth, the star-like structure also changes its position and shape across the *u*-*v* plane, depending on the declination of the source.





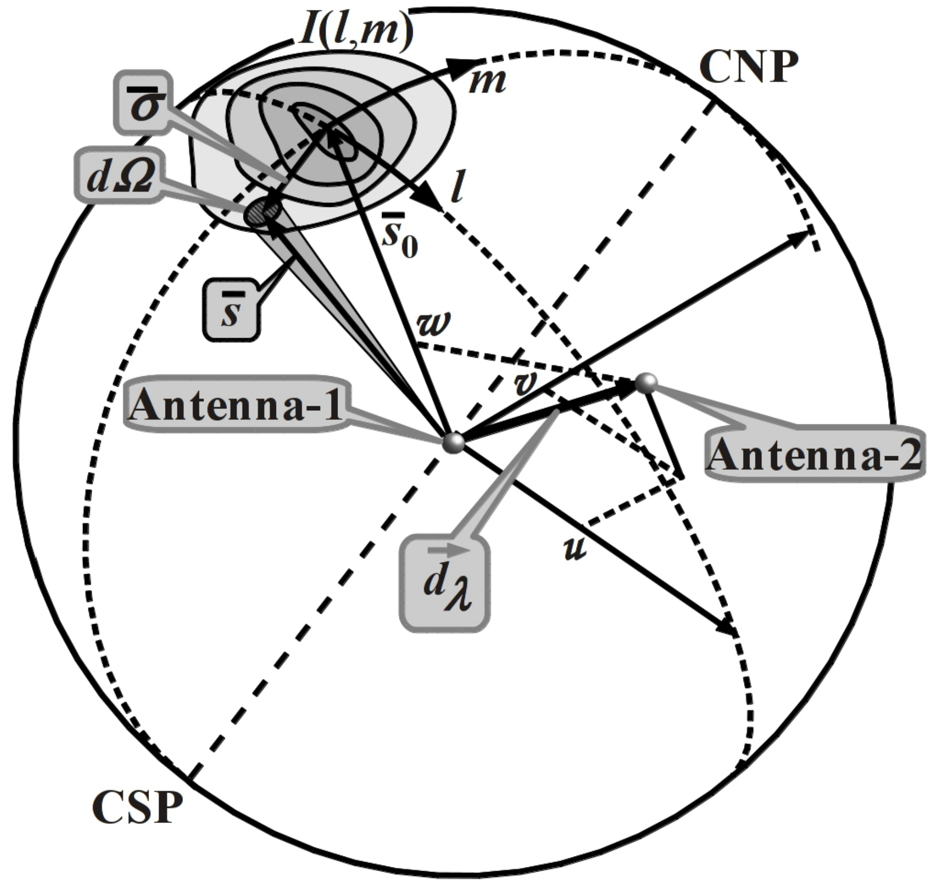
Left shows a typical *u*-*v* plane coverage obtained in few hours by tracking a source with changing declination. The VLA antennas are set on rail tracks. Hence after a twelve hour period, the antennas can be repositioned for new unique baselines and the observation can be repeated. In this way the *u*-*v* plane may be populated more and more.

Interferometry Radio Mapping

The next step after filling the synthesized aperture with visibility data is to generate an image of the radio source. The procedure involves a mapping of the visibilities in the u-v plane with source intensity distribution $I(l,m)$ on the sky (l-m plane). The l,m coordinates are in spatial domain, whereas the u,v coordinates are in spatial frequency domain. Intuitively we find a Fourier transform like relationship between the two. We have already talked about the basis of radio mapping between the source intensity and the aperture electric field of a single dish radio telescope in lecture-2 and lecture-7. The mapping between the source intensity distribution $I(l,m)$ and the visibilities $\mathcal{V}(u,v,w)$ occurs through van Cittert-Zernike equation which we have touched in lecture-2. We now derive this equation for a detailed understanding of the mapping procedure. For this, we first establish a relationship between visibility and correlation, which has been mentioned in lecture-2.

Visibility-Correlation Relationship-I

The visibility $\mathcal{V}(u,v,w)$ or the spatial coherence function are computed from the values of cross-correlations $r(\tau_g)$ obtained from the correlator. Visibilities are actually scaled values of the cross-correlations which are functions of the u,v,w coordinates. In this derivation we use a correlator with no delay compensation since we aim to establish a basic relationship between visibility and correlation. In other words, we consider the cross correlation r as a function of geometric delay τ_g.



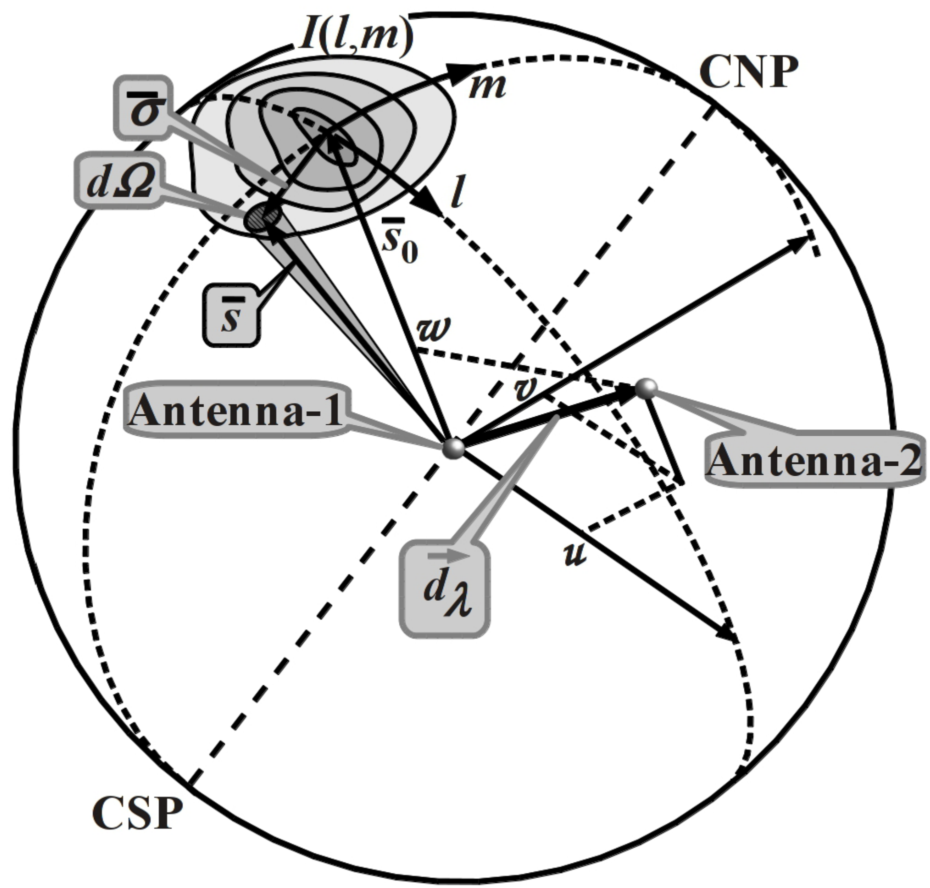
Consider an interferometer having two antennas observing a spatially incoherent radio source on the celestial sphere. Antenna-1 is positioned at the origin of the u,v,w coordinate system. The intensity distribution of the source is represented as $I(l,m)$. The origin of the l,m coordinate system is located on the phase reference position, usually at center of the source. It is also known as ***phase tracking center***.

Visibility-Correlation Relationship-II

$\bar{s}_0$ - Space vector pointed to the phase tracking center from antenna-1.

$\bar{\sigma}$ - Any nearby position on the sky with reference to the phase tracking center.

$\bar{s}$ - Space vector pointed to any point on the source field from antenna-1.



The above three vectors are related as:

$$\bar{s} = \bar{s}_0 + \bar{\sigma}$$

Radio intensity distribution of source on the celestial sphere: $I(l, m)$

Sky intensity/brightness at frequency ν in the direction $\bar{s}$: $I(\bar{s})$

Effective aperture area of an antenna in same direction: $A_e(\bar{s})$

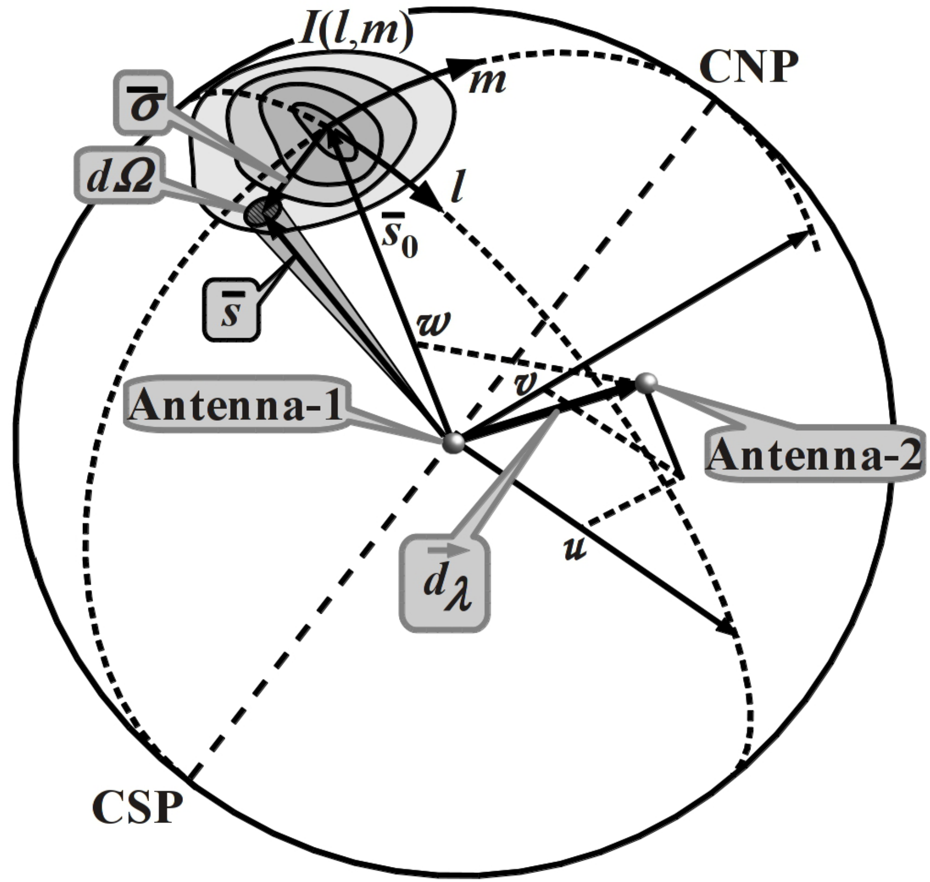
Over a solid angular element $d\Omega$ along $\bar{s}$, the signal power received over a band-width $\Delta\nu$ is: $A_e(\bar{s})\, I(\bar{s})\, \Delta\nu\, d\Omega$ ... (1)

Hence, an infinitesimal correlated signal power dr (per solid angle $d\Omega$) is given as: $dr = A_e(\bar{s})\, I(\bar{s})\, \Delta\nu\, d\Omega\, \cos(2\pi\nu\tau_g)$ ... (2)

Visibility-Correlation Relationship-III

Since the antennas have three dimensional patterns, the correlator power r can be obtained by integrating Eq. (2) over the celestial sphere as:

$$r(\vec{d_\lambda}, \bar{s}) = \Delta\nu \int_{-\infty}^{\infty} A_e(\bar{s})\, I(\bar{s})\, \cos\left[2\pi(\vec{d_\lambda}\cdot\bar{s})\right]\, d\Omega \quad \ldots (3)$$



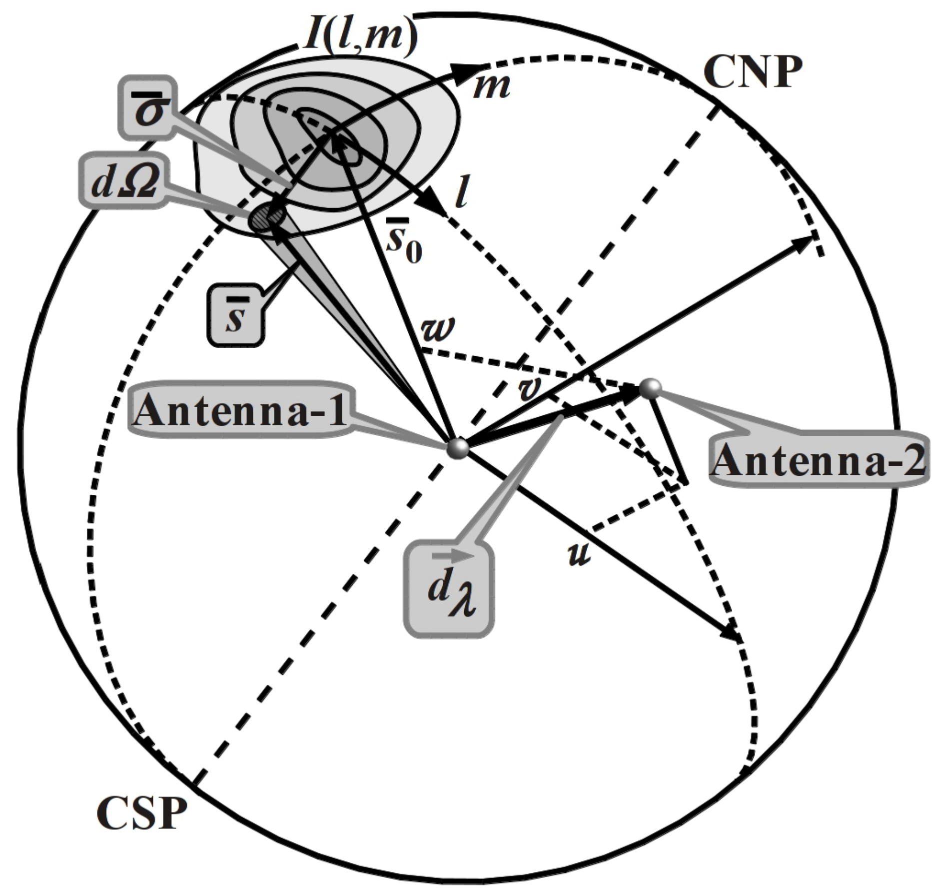
Here, $\vec{d_\lambda}$ represents the baseline space vector (in number of wavelengths) pointing from antenna-1 to antenna-2.

Since the antenna beam-widths are extremely small, only regions very close to the phase reference position are relevant. Hence we make measurements with respect to $\bar{s}_0$. Thus we express r in terms of $\bar{\sigma}$ and $\bar{s}_0$ as:

$$r(\vec{d_\lambda}, \bar{s}_0) = \Delta\nu\, \cos\left[2\pi(\vec{d_\lambda}\cdot\bar{s}_0)\right] \int_{-\infty}^{\infty} A_e(\bar{\sigma})\, I(\bar{\sigma})\, \cos\left[2\pi(\vec{d_\lambda}\cdot\bar{\sigma})\right]\, d\Omega$$
$$-\Delta\nu\, \sin\left[2\pi(\vec{d_\lambda}\cdot\bar{s}_0)\right] \int_{-\infty}^{\infty} A_e(\bar{\sigma})\, I(\bar{\sigma})\, \sin\left[2\pi(\vec{d_\lambda}\cdot\bar{\sigma})\right]\, d\Omega \quad \ldots (4)$$

Visibility-Correlation Relationship-IV

Since the source is spatially incoherent, which is applicable to all cosmic radio sources, the radiated waveforms from different elements of the source within $d\Omega$ are uncorrelated.



A normalized antenna beam pattern P_n is obtained as:

$$P_n(\bar{\sigma}) = A_e(\bar{\sigma})/A_{e0} \quad \ldots (5)$$

Here A_{e0} is the maximum effective aperture area of the antenna which is along the the direction of the radio source since antennas are pointed towards it.

Since the visibility $\mathcal{V}$ is a complex function, it can be expressed in relation to the antenna pattern as:

$$\mathcal{V} = |\mathcal{V}|\, e^{j\phi_v} = \int_{-\infty}^{\infty} P_n(\bar{\sigma})\, I(\bar{\sigma})\, e^{-j2\pi \vec{d}_\lambda \cdot \bar{\sigma}}\, d\Omega \quad \ldots (6)$$

Here, φ_v is the phase of visibility.

Visibility-Correlation Relationship-V

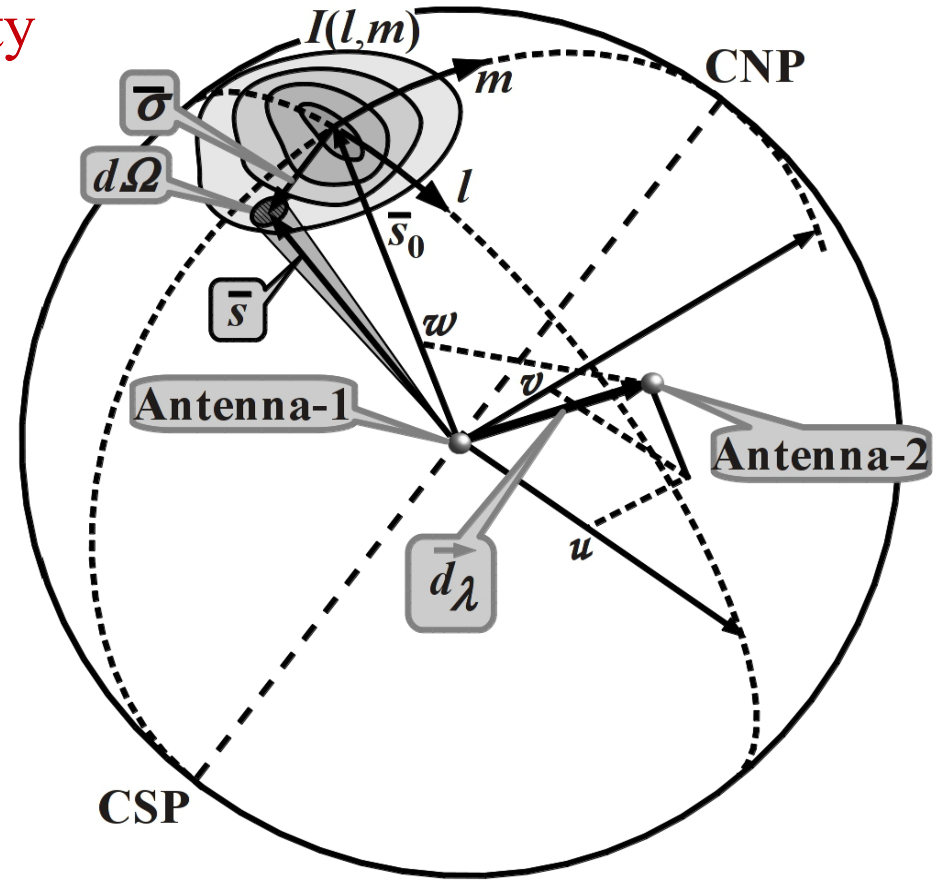
The real and imaginary parts of the visibility can now be separately shown as:

$$|\mathcal{V}|\cos\phi_v = \int_{-\infty}^{\infty} P_n(\bar{\sigma})\, I(\bar{\sigma})\, \cos(2\pi \vec{d_\lambda}\cdot\bar{\sigma})\, d\Omega \quad \ldots (7)$$

$$|\mathcal{V}|\sin\phi_v = -\int_{-\infty}^{\infty} P_n(\bar{\sigma})\, I(\bar{\sigma})\, \sin(2\pi \vec{d_\lambda}\cdot\bar{\sigma})\, d\Omega \quad \ldots (8)$$

Relating the above two equations with Eq. (4), we finally establish a relationship between visibility and correlation as:

$$r(\vec{d_\lambda}, \bar{s}_0) = A_{e0}\,\Delta\nu\, |\mathcal{V}| \cos\left[2\pi(\vec{d_\lambda}\cdot\bar{s}_0 - \phi_v)\right] \quad \ldots (9)$$



It indicates that the interferometer is an instrument which measures the visibility, which is the spatial coherence function having a different normalization.

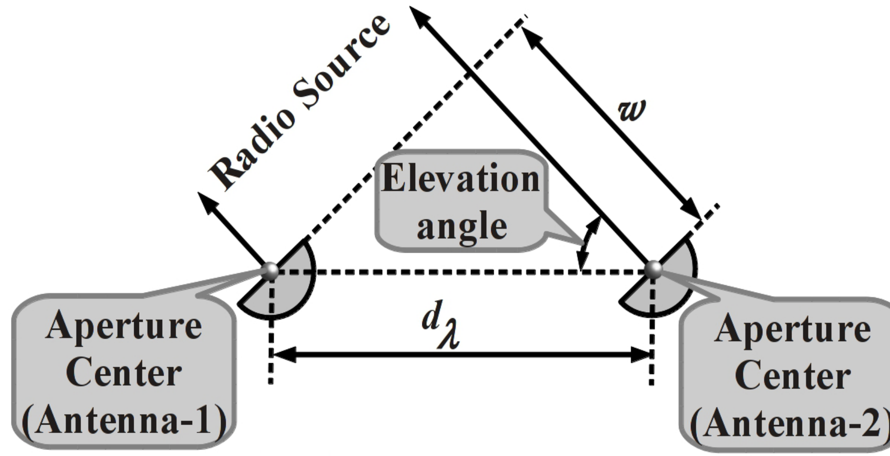
Van Cittert Zernike Equation-I

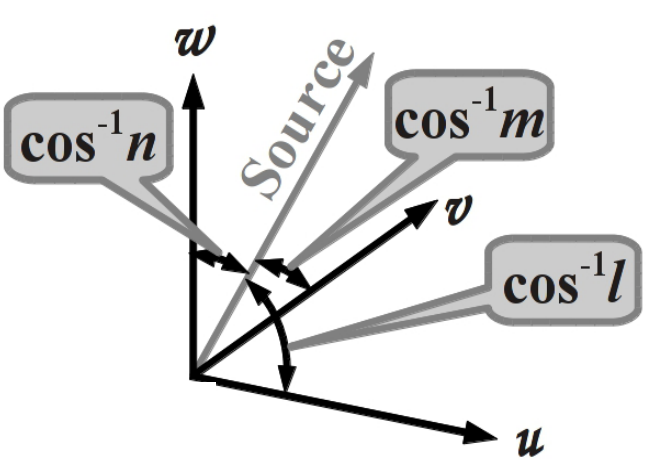
In last section, we established a relationship between visibility and correlation as:

$$\boxed{r(\vec{d_\lambda}, \bar{s}_0) = A_{e0}\, \Delta\nu\ |\mathcal{V}| \cos\left[2\pi(\vec{d_\lambda}\cdot\bar{s}_0 - \phi_v)\right]} \quad \ldots (1)$$

It is now necessary to know the values of visibilities and their positions on the u,v,w coordinates for making a radio image in l,m coordinates.

Hence Eq. (1) needs to be related with the u,v,w and the l,m,n coordinate system. As shown on right, with decreasing elevation angle, the value of w increases.





A point on the celestial sphere are defined using l, m and n axes which are direction cosines related to u, v and w axes respectively as shown on left. A synthesized image on the l-m surface is therefore a projection of that region of celestial sphere onto a plane which is tangent to the origin of l,m coordinates.

Van Cittert Zernike Equation-II

Various relations among baseline vector (in number of wavelengths), point of observation, u,v,w coordinates and l,m,n coordinates are as follows:

$$\vec{d_\lambda} \cdot \bar{s} = ul + vm + wn \quad \ldots (2)$$

$$\vec{d_\lambda} \cdot \bar{s_0} = w \quad \ldots (3)$$

$$d\Omega = \frac{dl\, dm}{n} = \frac{dl\, dm}{\sqrt{1 - l^2 - m^2}} \quad \ldots (4)$$

We can now express the visibility $\mathcal{V}$ obtained from Eq. (1) as a function of u,v,w as shown below, which is known as the van Cittert-Zernike equation.

$$\boxed{\mathcal{V}(u,v,w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{P_n(l,m)\, I(l,m)}{\sqrt{1-l^2-m^2}}\, e^{-j2\pi\left[ul+vm+w\left(\sqrt{1-l^2-m^2}-1\right)\right]} dl\, dm} \quad \ldots (5)$$

The van Cittert-Zernike equation proves that a complex visibility function $\mathcal{V}(u,v,w)$ is a Fourier like integral of the sky brightness $I(l,m)$, multiplied by the primary beam response of the interferometer, $P_n(l,m)$ and a factor of $1/(1 - l^2 - m^2)^{1/2}$. The u,v,w coordinate system have been defined in such a way that the w-axis always points towards the radio source.

Van Cittert Zernike Equation-III

When the source extent is small, the quantities $|l|$ and $|m|$ are also small. Hence the term $w[(1 - l^2 - m^2)^{½} - 1]$ in the exponent becomes $-½(l^2 + m^2)w$ which is very small and therefore neglected. With this simplification, Eq. (5) becomes:

$$\boxed{\mathcal{V}(u,v,w) \simeq \mathcal{V}(u,v,0) = \int_{-l_1}^{l_2}\int_{-m_1}^{m_2} \frac{P_n(l,m)\, I(l,m)}{\sqrt{1-l^2-m^2}}\, e^{-j2\pi(ul+vm)} dl\, dm} \quad \ldots (6)$$

Hence, within a small range of l (l_1, l_2) and m (m_1,m_2), the visibility $\mathcal{V}(u,v,w)$ can be thought to be independent of w. Taking the inverse Fourier transform of Eq. (6) and after some re-arrangements, we get an expression of $I(l,m)$ in terms of visibility as:

$$I(l,m) = \frac{\sqrt{1-l^2-m^2}}{P_n(l,m)} \int_{-l_1}^{l_2}\int_{-m_1}^{m_2} \mathcal{V}(u,v)\, e^{j2\pi(ul+vm)} du\, dv \quad \ldots (7)$$

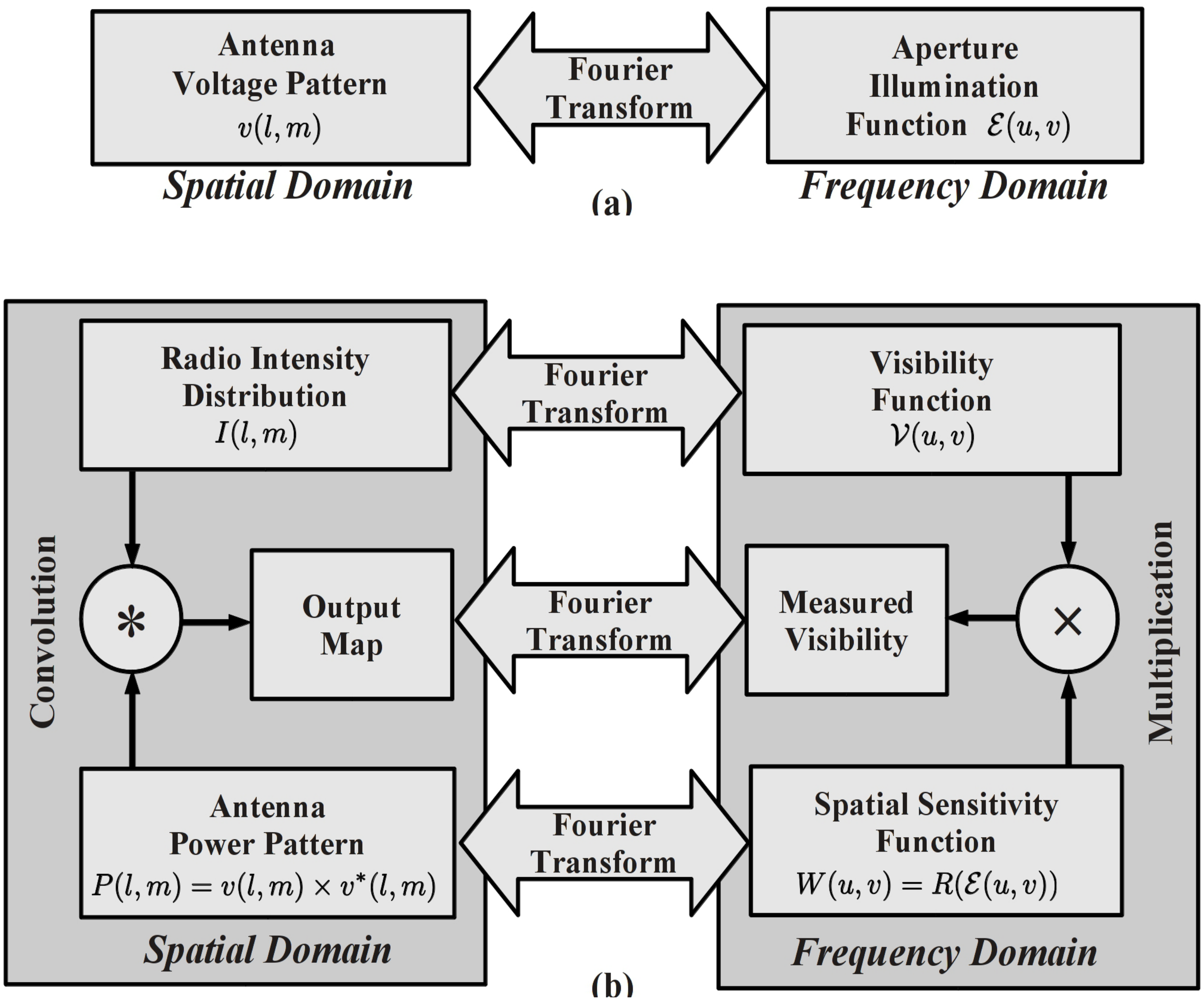
Assuming the antenna beam-widths are limited to (l_1, l_2) and (m_1,m_2), we may extend the limits of integration to $\pm\infty$ since the results are unchanged. Hence Eq. (7) can be re-written as:

$$\boxed{I(l,m) = \frac{\sqrt{1-l^2-m^2}}{P_n(l,m)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{V}(u,v)\, e^{j2\pi(ul+vm)} du\, dv} \quad \ldots (8)$$

Van Cittert Zernike Equation-IV

Eq. (8) constructs radio intensity distribution maps of the source, provided the source is small enough to be accommodated within the primary beam pattern.

The measured visibility is different from the true visibility due to convolution of intensity distribution with the beam pattern. The power pattern $P(l,m)$ of the antenna is the product of antenna voltage pattern $v(l,m)$ with its conjugate $v^*(l,m)$ in the spatial domain. The spatial sensitivity function $W(u,v)$ is obtained from the auto-correlation of the aperture illumination function $\mathcal{E}(u,v)$ in the frequency domain.



Assignment Problems-I

1. Describe the basic technique for synthesizing a large aperture using two small aperture antennas when the source is at the zenith.

2. If the synthesized aperture required is 100 m^2 for a source at zenith, how many locations the second antenna must be positioned, given the aperture size of the antenna elements as 2 m^2 ?

3. Do we really require two antennas for aperture synthesis? What is the difficulty in doing so using one antenna? Justify your answer using equations. Hint: $F = v^2 e^{j(\phi_m - \phi_n)}$

4. What do you understand by "Earth rotation super-synthesis"?

5. Two antennas are positioned at a latitude of 45° North on Earth which observes as source on the sky. Let a rectangular coordinate system (x,y,z) be originated at the center of the Earth with its z-axis always pointed to the source. Using diagrams illustrate the projections of the antennas in the x-y plane as seen from the source for the cases of source located at (*i*) CNP, (*ii*) celestial equator, and (*iii*) a celestial latitude of 45° North.

Assignment Problems-II

7. For a single dish having a diameter D operated at a radio wavelength , what is range of spatial frequencies this single dish covers ?

Ans: 0 to $D/2$

8. While observing a source above the celestial equator for twelve hours, an interferometer generates visibilities $\mathcal{V}(u,v)$ whose locus on the u-v plane is half an ellipse. Using the same data how can you create the other half of the ellipse?

Ans: $\mathcal{V}(-u,-v) = \mathcal{V}^*(u,v)$

9. An interferometer situated along the East-West line observes a radio source. The visibilities obtained are spread across the u-v plane. Let the wavelength be λ, distance between the antennas be d, and the source makes an angle θ with the CNP. At any instant of time what is distance from the center of the u-v coordinates at which the visibility data will be lie?

Hint: $a = \dfrac{d}{\lambda}$ $\quad b = \dfrac{d\ \cos\theta}{\lambda} = \dfrac{d\ \sin\delta}{\lambda}$

Assignment Problems-III

10. An East-West interferometer tracks a radio source. Unless the source is on the celestial equator, the traces of visibilities on the u-v plane varies from ellipse to a circle (at the celestial pole). What happens if the antennas are along North-South directions ? Explain in details.

11. Explain the advantages of building an interferometer array having baselines components along North-South directions? Give an examples of such an arrays.
Hint: Radio images can be made even for sources located on the celestial equator.

12. Explain the relation between visibility and correlation using an equation.
Hint:
$$r(\vec{d_\lambda}, \bar{s}_0) = A_{e0}\,\Delta\nu\;|\mathcal{V}|\cos\left[2\pi(\vec{d_\lambda}\cdot\bar{s}_0 - \phi_v)\right]$$

13. State and explain the significance of the van Cittert-Zernike equation.
Hint:
$$\mathcal{V}(u,v,w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{P_n(l,m)\,I(l,m)}{\sqrt{1-l^2-m^2}}\,e^{-j2\pi\left[ul+vm+w\left(\sqrt{1-l^2-m^2}-1\right)\right]}dl\,dm$$

Assignment Problems-IV

14. With help of a block diagram illustrate the relationships existing between (*i*) radio intensity distribution, (*ii*) output map, (*iii*) antenna power pattern, (*iv*) visibility function, (*v*) measured visibility, and (*vi*) spatial sensitivity function.

THANK YOU